

# On the nonexplosion and explosion for nonhomogeneous Markov pure jump processes in Borel spaces

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**Abstract:** In this paper, we obtain new drift-type conditions for nonexplosion and explosion for nonhomogeneous Markov pure jump processes in Borel state spaces. The conditions are practical and in the meanwhile sharp; e.g., the one for nonexplosion is necessary if the state space is in addition locally compact and the  $Q$ -function satisfies weak Feller-type and local boundedness conditions. We comment on the relations of our conditions with the existing ones in the literature, and demonstrate some possible applications.

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## 1 Introduction

In this paper we obtain conditions for nonexplosion and explosion of nonhomogeneous pure jump Markov processes in Borel state spaces. More precisely, we consider the transition rate or say the  $Q$ -function (or  $Q$ -matrix in case the state space is denumerable) to be conservative and stable. According to Feller [13], see also [12], for each  $Q$ -function, there is a (minimal) transition function solving the Kolmogorov backward equation, giving rise to a (minimal) nonhomogeneous Markov pure jump process. It is this process that is of our interest here. The relevant definitions will be given in the next section.

Markov pure jump processes provide a useful tool for modeling in queueing systems, telecommunication, inventory, and stochastic particle systems etc, and is also the foundation in the theory of continuous-time Markov decision processes. In the context of stochastic particle systems, one often needs consider the process to be in an uncountable state space [10, 32], whereas in the context of continuous-time Markov decision processes, for the sake of optimality in general one needs deal with a nonstationary strategy inducing a nonhomogeneous Markov pure jump process. A stochastic particle system is said to exist if the underlying Markov pure jump process is nonexplosive, see e.g., [10].

The study of conditions for nonexplosion of Markov pure jump processes has been done by many authors for many years but mainly for the homogeneous denumerable case, i.e., when the  $Q$ -matrix does not depend on the time, and the state space is denumerable; such a Markov pure jump process is called a continuous-time Markov chain. A desirable condition would be sharp and at the same time practically applicable. In this connection, for continuous-time Markov chains, Reuter [28] in 1950s obtained the first condition necessary and sufficient for nonexplosion which is in the meanwhile exclusively in terms of the  $Q$ -matrix. That condition can be stated as the absence of nontrivial bounded nonnegative eigenvectors of the  $Q$ -matrix for any positive eigenvalue. Clearly, in practice, it

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is not easy to verify the Reuter's condition. In response to this, in 1980s Mufa Chen [4] obtained a verifiable condition for nonexplosion in terms of the existence of a drift function with respect to the  $Q$ -function. The sufficiency of this condition was justified for the homogeneous but general state space case in [4, 6]. Very recently, Spieksma [31] showed that Chen's condition in [4] is actually necessary for the continuous-time Markov chain. In the survey by Mufa Chen [8], the necessity of that condition in the homogeneous but general state space case was said to be unclear, see the proof of Theorem 4 as well as the last paragraph of the text therein. (In the present paper the necessity of Chen's condition is justified if the state space is locally compact separable and the  $Q$ -function satisfies some weak-Feller type conditions.) For the other relevant results for the homogeneous case, we refer the interested reader to the book [1] and the recent survey [8] in case the state space is denumerable, and the monograph [6] in the case of a more general state space. See also [22] for some more recent progresses, where more specific continuous-time Markov chains were considered.

Less is known for the nonhomogeneous case. In case the state space is denumerable, and in addition the  $Q$ -function is continuous in time, a drift condition for the nonexplosion was independently obtained by Chow and Khasminskii [9] recently, see Theorem 1 therein, and Junli Zheng and Xiaogu Zheng [35, 36]. The similar conditions were obtained in the context of continuous-time Markov decision processes in Borel spaces, where each Markov strategy induces a nonhomogeneous Markov pure jump process, see e.g., [16, 18]; see also the relevant reference in [25]. Specialised to the denumerable case with a continuous  $Q$ -function, the conditions known in [16, 18, 25] are stronger than the one in [9, 35]. In Meyn and Tweedie [24], a sufficient condition for the nonexplosion of the general continuous-time Markov process in locally compact separable metric state spaces was obtained, although it is in terms of the extended generator of the process, and assumed apriori that the test function is in the domain of the extended generator; this fact is not always trivial, and so the condition in [24] is not directly applicable in practice. No discussions were provided in [9, 16, 18, 24, 25, 35, 36] on the sharpness of their conditions; recall the aforementioned remark on this left in the survey [8]. To the best of our knowledge, we seem not to be aware of other practically applicable conditions in the literature for the nonexplosion of general nonhomogeneous Markov pure jump processes in Borel state spaces.

The main contribution of the present paper is that for nonhomogeneous Markov pure jump processes in Borel state spaces we obtain new and sharp drift-type conditions for nonexplosion, where the sharpness is understood in the sense that our condition is necessary in case the state space is locally compact separable metrizable and some weak Feller-type and local boundedness conditions are satisfied; this is in particular the case if the state space is denumerable. As an application of this condition, a generalization of the condition in Chow and Khasminskii [9] and Junli Zheng and Xiaogu Zheng [35, 36] is obtained for the general Borel state space case, where the continuity of the  $Q$ -function is removed. Our arguments are different from those of [9, 16, 18, 24, 25, 35, 36], which, roughly speaking, are based on the study of a sequence of truncated continuous-time models with the truncation being done on the state space. The arguments here are based on the connection between the nonexplosion of the continuous-time process and the stability and optimal control of an underlying discrete-time Markov process, and are extensions of the one in the interesting work of Spieksma [31], which deals with the homogeneous denumerable case.

The rest of this paper is organized as follows. Basic notations and definitions about nonhomogeneous Markov pure jump processes are summarized in Section 2. Our conditions for nonexplosion and explosion along with their relation to the existing ones are presented in Section 3. We demonstrate possible applications of the conditions for nonexplosion to the stability of (homogeneous) Markov pure jump processes in Borel spaces, and the continuous-time Markov decision processes with nonstationary strategies in Section 4. We finish the paper with a conclusion in Section 5. The auxiliary statements and proofs are postponed to the appendix.

## 2 Definitions, notations and basic facts

In what follows, let  $S$  be a nonempty Borel state space endowed with its Borel  $\sigma$ -algebra  $\mathcal{B}(S)$ . The term “measurability” is always understood in the Borel sense. The Dirac measure concentrated on the singleton  $\{x\}$  is denoted as  $\delta_x(dy)$ .

As in Feinberg, Mandava and Shiryaev [12], we adopt the following definition throughout this paper.

**Definition 2.1** *A (Borel-measurable) signed kernel  $q(dy|x, s)$  on  $\mathcal{B}(S)$  from  $S \times [0, \infty)$  is called a (conservative stable)  $Q$ -function on the Borel space  $S$  if the following conditions are satisfied.*

(a) For each  $s \geq 0$ ,  $x \in S$  and  $\Gamma \in \mathcal{B}(S)$  with  $x \notin \Gamma$ ,

$$\infty > q(\Gamma|x, s) \geq 0.$$

(b) For each  $(x, s) \in S \times [0, \infty)$ ,

$$q(S|x, s) = 0.$$

(c) For each  $x \in S$ ,

$$\sup_{s \geq 0} \{q(S \setminus \{x\}|x, s)\} < \infty.$$

Given a  $Q$ -function  $q$  on  $S$ , we introduce the kernel  $\tilde{q}(dy|x, s)$  on  $\mathcal{B}(S)$  from  $S \times [0, \infty)$  by

$$\tilde{q}(\Gamma|x, s) := q(\Gamma \setminus \{x\}|x, s)$$

for each  $x \in S$ ,  $s \in [0, \infty)$  and  $\Gamma \in \mathcal{B}(S)$ , and the function

$$q_x(s) := -q(\{x\}|x, s), \quad \forall x \in S, \quad s \in [0, \infty).$$

For each (conservative stable)  $Q$ -function  $q$  on a Borel space  $S$ , it is convenient to agree on that

$$q_x(\infty) := 0 =: \tilde{q}(\Gamma|x, \infty) \tag{1}$$

for each  $x \in S$  and  $\Gamma \in \mathcal{B}(S)$ .

For each  $\Gamma \in \mathcal{B}(S)$ ,  $x \in S$ ,  $s, t \in [0, \infty)$  and  $s \leq t$ , we define

$$\begin{aligned} P_q^{(0)}(s, x, t, \Gamma) &:= \delta_x(\Gamma) e^{-\int_s^t q_x(v) dv}, \\ P_q^{(n+1)}(s, x, t, \Gamma) &:= \int_s^t e^{-\int_s^u q_x(v) dv} \left( \int_S P_q^{(n)}(u, z, t, \Gamma) \tilde{q}(dz|x, u) \right) du, \quad \forall n = 0, 1, \dots \end{aligned} \tag{2}$$

It is well known or otherwise easy to check that the following assertions hold;

- (a) for each  $s, t \in [0, \infty)$ ,  $s \leq t$  and  $x \in S$ ,  $P_q^{(n)}(s, x, t, dy)$  is a  $[0, 1]$ -valued measure on  $\mathcal{B}(S)$ ;
- (b) for each  $\Gamma \in \mathcal{B}(S)$ ,  $(s, x, t) \rightarrow P_q^{(n)}(s, x, t, \Gamma)$  is jointly measurable in  $x \in S$  and  $s, t \in [0, \infty)$ ,  $s \leq t$ .

Define

$$P_q(s, x, t, \Gamma) := \sum_{n=0}^{\infty} P_q^{(n)}(s, x, t, \Gamma) \quad (3)$$

for each  $x \in S$ ,  $s, t \in [0, \infty)$ ,  $s \leq t$ , and  $\Gamma \in \mathcal{B}(S)$ .

The following result is due to Feller [13], where the extra condition on the continuity of the  $Q$ -function can be withdrawn; see also [12, 33].

**Proposition 2.1** (a) For each  $s, t \in [0, \infty)$ ,  $s \leq t$  and  $x \in S$ ,  $P_q(s, x, t, dy)$  is a  $[0, 1]$ -valued measure on  $\mathcal{B}(S)$ .

(b) For each  $\Gamma \in \mathcal{B}(S)$ ,  $(s, x, t) \rightarrow P_q(s, x, t, \Gamma)$  is measurable (jointly) in  $x \in S$  and  $s, t \in [0, \infty)$ ,  $s \leq t$ .

(Thus, from (a) and (b),  $P_q(s, x, t, dy)$  is a transition function.)

(c) The transition function  $P_q(s, x, t, dy)$  satisfies the following Chapman-Kolmogorov equation;

$$P_q(s, x, t, \Gamma) = \int_S P_q(s, x, u, dy) P_q(u, y, t, \Gamma), \quad \forall x \in S, s, u, t \in [0, \infty), s < u < t, \Gamma \in \mathcal{B}(S).$$

Throughout this paper, the transition function  $P_q(s, x, t, dy)$  defined by (3) is called the  $q$ -transition function (to signify its dependence on the  $Q$ -function  $q$ ). In the literature, it is also called the Feller's minimal  $q$ -transition function or  $q$ -process, where the minimality is in the sense that it is the minimal nonnegative solution to the Kolmogorov backward equation; see Theorem 3.2 of [12]. The later fact is not needed in this paper, whereas the following Kolmogorov forward equation will be used.

**Proposition 2.2** For each  $x \in S$ ,  $s, t \in [0, \infty)$ ,  $s \leq t$  and  $\Gamma \in \mathcal{B}(S)$  satisfying

$$\sup_{x \in \Gamma, s \geq 0} q_x(s) < \infty, \quad (4)$$

the following holds;

$$\begin{aligned} P_q(s, x, t, \Gamma) &= \delta_x(\Gamma) + \int_s^t \left( \int_S \tilde{q}(\Gamma|y, u) P_q(s, x, u, dy) \right) du - \int_s^t \left( \int_{\Gamma} q_y(u) P_q(s, x, u, dy) \right) du \\ &= \delta_x(\Gamma) + \int_s^t \left( \int_S q(\Gamma|y, u) P_q(s, x, u, dy) \right) du. \end{aligned} \quad (5)$$

*Proof.* See Corollary 4.2 of [12]. The second equality holds because of (4).  $\square$

It is worthwhile to point out that in general (5) does not hold for all  $\Gamma \in \mathcal{B}(S)$ ; the statements of Lemma 1(ii) and Theorem 2(ii) of [33] are not quite accurate.

For the subsequent discussions, the connection between the  $q$ -transition function  $P_q(s, x, t, dy)$  and a (nonhomogeneous) pure jump Markov process is briefly described as follows. (The term of nonhomogeneous is traditional, and it is understood as not necessarily homogeneous.)

Let  $x_\infty \notin S$  be some fixed isolated point. Denote  $S_\infty := S \cup \{x_\infty\}$ . Let the sample space  $\Omega$  be the collection of sequences  $(x_0, t_1, x_1, t_2, \dots) \in (S_\infty \times (0, \infty])^\infty$  such that

(a)  $x_0 \in S$ , and

(b)  $\theta_n = \infty$  for some  $n = 1, 2, \dots$ , if and only if  $x_m = x_\infty$  and  $\theta_{m+1} = \infty$  for each  $m = n, n+1, \dots$ .

The  $\sigma$ -algebra  $\mathcal{F}$  on  $\Omega$  is the trace of the product  $\sigma$ -algebra of  $(S_\infty \times (0, \infty])^\infty$ . The generic notation  $\omega \in \Omega$  is in use but typically omitted; so that in what follows, without special reference the coordinate variables are often written as

$$x_n(\omega) = x_n, \quad \theta_{n+1}(\omega) = \theta_{n+1}.$$

(The abuse of notation here is conventional.) We define

$$t_n := \sum_{i=1}^n \theta_i, \quad \forall n = 1, 2, \dots, \quad t_0 := 0.$$

The accumulation point of the monotone nondecreasing sequence  $(t_n)$  is denoted by  $t_\infty$ .

The collection  $(t_n, x_n)$  is a marked point process. Let the filtration  $\{\mathcal{F}_t\}$  be the internal history of  $(t_n, x_n)$ . It is known, see e.g., [20] or Chapter 4 of [21], that for each fixed initial distribution  $\gamma$  on  $\mathcal{B}(S)$ , there is a unique probability  $\mathbb{P}_\gamma^q$  on  $(\Omega, \mathcal{F})$  such that the following assertions hold.

- (a)  $\mathbb{P}_\gamma^q(x_0(\omega) \in \Gamma) = \gamma(\Gamma)$  for each  $\Gamma \in \mathcal{B}(S)$ .
- (b) The random measure on  $\mathcal{B}((0, \infty) \times S)$  defined by

$$\nu(\omega, dt \times dx) := \sum_{n=0}^{\infty} I\{t_n < t \leq t_{n+1}\} \tilde{q}(dx|x_n, t) dt$$

is the dual predictable projection of the random measure of the marked point process  $(t_n, x_n)$ .

If the initial distribution  $\gamma$  is concentrated on a singleton say  $\{x\} \subseteq S$ , then  $\mathbb{P}_\gamma^q$  is written as  $\mathbb{P}_x^q$ . The expectation with respect to  $\mathbb{P}_\gamma^q$  and  $\mathbb{P}_x^q$  are denoted by  $\mathbb{E}_\gamma^q$  and  $\mathbb{E}_x^q$ . When it is not important to emphasize on the initial distribution, we simply write  $\mathbb{P}^q$  and  $\mathbb{E}^q$ .

Now we define the process of interest  $(X_t, t \geq 0)$  on  $(\Omega, \mathcal{F})$  with the filtration  $\{\mathcal{F}_t\}$  by

$$X_t(\omega) = \begin{cases} x_n, & \text{if } t_n \leq t < t_{n+1}, \\ x_\infty, & \text{if } t_\infty \leq t, \end{cases} \quad (6)$$

for each  $\omega \in \Omega$ .

**Proposition 2.3** *For each  $u, t \in (0, \infty)$ ,  $u \leq t$  and  $\Gamma \in \mathcal{B}(S)$ ,*

$$\mathbb{P}^q(X_t \in \Gamma | \mathcal{F}_u) = \mathbb{P}^q(X_t \in \Gamma | X_u) = P_q(u, X_u, t, \Gamma)$$

*almost surely with respect to  $\mathbb{P}^q$  on  $\{u < t_\infty\}$ .*

*Proof.* See Theorem 2.2 of [12]. □

Thus,  $(X_t)$  is a (nonhomogeneous) pure jump Markov process with the transition function  $P_q$ .

**Definition 2.2** *Consider the  $Q$ -function  $q$  on  $S$ . The (nonhomogeneous) pure jump process  $(X_t)$  with the transition function  $P_q$  is called nonexplosive if*

$$\mathbb{P}_x^q(t_\infty = \infty) = 1, \quad \forall x \in S.$$

*If for some  $x \in S$  such that  $\mathbb{P}_x^q(t_\infty = \infty) < 1$ , then the process  $(X_t)$  is called explosive.*

An objective of this paper is to obtain new, sharp and practical conditions for nonhomogeneous Markov pure jump processes. The arguments, different from those in e.g., [9, 24, 35, 36], are simple and transparent yet quite fruitful, which naturally lead to some known conditions in the previous literature. See more discussions and details in the next section. To this end, we firstly present the next simple but useful observation concerning the equivalent definitions of nonexplosion.

**Lemma 2.1** *Consider the  $Q$ -function  $q$  on  $S$ . Then the following assertions are equivalent.*

- (a) *The (nonhomogeneous) pure jump process  $(X_t)$  with the transition function  $P_q$  is nonexplosive.*
- (b) *For each  $x \in S$  and  $t \geq 0$ ,  $P_q(0, x, t, S) = 1$ .*
- (c) *For each  $x \in S$ , and almost all  $t \geq 0$ ,  $P_q(0, x, t, S) = 1$ .*
- (d) *For each  $x \in S$  and  $0 \leq s \leq t$ ,  $P_q(s, x, t, S) = 1$ .*

*Proof.* See the appendix. □

To finish this section, for a  $Q$ -function  $q$  on (the Borel space)  $S$  and a constant  $c \in (0, \infty)$ , let us introduce the  $f$ -transformed  $Q$ -function as follows, where  $f$  is a  $(0, \infty)$ -valued measurable function on  $S$  satisfying

$$\int_S f(y) \tilde{q}(dy|x, s) < \infty, \quad \int_S f(y) q(dy|x, s) \leq cf(x) \quad \forall x \in S, s \geq 0.$$

Let

$$S_\delta := S \cup \{\delta\}$$

with  $\delta \notin S$  being an isolated point and satisfies  $\delta \neq x_\infty$ . (Here we choose a different isolated point because we want to reserve  $x_\infty$  as the isolated point in the definition of  $(X_t)$ , see (6), when the state space is  $S_\delta$ .) We define the (conservative and stable)  $Q$ -function  $q^f$  on  $S_\delta$  (instead of  $S$ ) by

$$q^f(\Gamma|x, s) = \begin{cases} \frac{\int_\Gamma f(y) q(dy|x, s)}{f(x)}, & \text{if } x \in S, \Gamma \in \mathcal{B}(S), x \notin \Gamma; \\ c - \frac{\int_S f(y) q(dy|x, s)}{f(x)}, & \text{if } x \in S, \Gamma = \{\delta\}; \\ 0, & \text{if } x = \delta, \Gamma = S. \end{cases} \quad (7)$$

for each  $s \geq 0$ . Therefore, the isolated state  $\delta$  is absorbing, and the transition rate at  $x \in S$  is given by

$$q_x^f(s) = c + q_x(s), \quad \forall s \geq 0.$$

Below we reserve the isolated point  $\delta$  only for the  $f$ -transformed model. The  $q^f$ -transition function on  $S_\delta$  is denoted by  $P_{q^f}(s, x, t, dy)$ . The nonhomogeneous pure jump Markov process with the transition function  $P_{q^f}$  is still denoted by  $(X_t)$  despite its state space is different now, but its corresponding probability and expectation will be signified with the superscript  $q^f$ .

The notion of the  $f$ -transformed function appeared in Anderson [1] and Spieksma [30] studying a homogeneous Markov pure jump process in a denumerable state space. The more specific form when  $f$  is a constant function has been long used in continuous-time Markov decision processes with discounted costs, see the next section. The relationship between the  $q$ -transition function and the  $q^f$ -transition function is given by Lemma A.3 in the appendix.

### 3 Explosion and nonexplosion

Consider a  $Q$ -function  $q$  on  $S$ . The objective of this section is to provide general, natural, practical yet sharp conditions for the nonexplosion and explosion of the the nonhomogeneous Markov pure jump process  $(X_t)$  in a Borel state space  $S$  with the transition function  $P_q$ . The simpler homogeneous case is commented at the end of this section.

It follows from Lemma 2.1 that the process  $(X_t)$  with the transition function  $P_q$  is nonexplosive if and only if for some constant  $\alpha \in (0, \infty)$ ,

$$\alpha \int_0^\infty e^{-\alpha t} P_q(v, x, v+t, S) dt = 1, \quad \forall x \in S, v \in [0, \infty). \quad (8)$$

Let  $\Delta \notin S$  be an isolated point such that  $\Delta \neq \delta$  and  $\Delta \neq x_\infty$ . Now consider the (homogeneous) discrete-time Markov chain  $(\tilde{t}_n, \tilde{x}_n)$  in the state space  $([0, \infty) \times S) \cup (\{\Delta\} \times \{\infty\})$  with the transition probability given by

$$p^\alpha(\Gamma_1 \times \Gamma_2 | v, x) := \begin{cases} \int_{\Gamma_1 \cap (v, \infty)} e^{-\int_0^{t-v} (\alpha + q_x(s+v)) ds} \tilde{q}(\Gamma_2 | x, t) dt, & \text{if } x \in S, v \in [0, \infty), \Gamma_1 \in \mathcal{B}([0, \infty)), \Gamma_2 \in \mathcal{B}(S); \\ 1 - \int_0^\infty e^{-\alpha t - \int_0^t q_x(s+v) ds} q_x(t+v) dt, & \text{if } x \in S, v \in [0, \infty), \Gamma_1 = \{\infty\}, \Gamma_2 = \{\Delta\}; \\ 1, & \text{if } x = \delta, v = \infty, \Gamma_1 = \{\infty\}, \Gamma_2 = \{\Delta\}. \end{cases} \quad (9)$$

(The transition probability on other states to other sets can be automatically calculated based on the above definitions.) Clearly, the state  $(\infty, \Delta)$  is an isolated (cemetery) point. Let  $\tilde{P}_{(v,x)}$  be the corresponding probability measure for this discrete-time Markov chain with the transition probability  $p^\alpha$  and the initial state  $(v, x) \in ([0, \infty) \times S) \cup (\{\infty\} \times \{\Delta\})$ .

As in [2, 14, 34], one can see that for each  $x \in S$  and  $v \in [0, \infty)$ ,

$$\begin{aligned} \alpha \int_0^\infty e^{-\alpha t} P_q(v, x, v+t, S) dt &= \tilde{E}_{(v,x)} \left[ \sum_{n=0}^\infty I\{\tilde{x}_n \in S\} \int_0^\infty \alpha e^{-\alpha t} e^{-\int_0^t q_{\tilde{x}_n}(s+\tilde{t}_n) ds} dt \right] \\ &= \tilde{E}_{(v,x)} \left[ \sum_{n=0}^\infty I\{\tilde{x}_n \in S\} \left( 1 - \int_0^\infty e^{-\alpha t - \int_0^t q_{\tilde{x}_n}(s+\tilde{t}_n) ds} q_{\tilde{x}_n}(\tilde{t}_n + t) dt \right) \right] \\ &= \tilde{E}_{(v,x)} \left[ \sum_{n=0}^\infty I\{\tilde{x}_n \in S\} p^\alpha(\{\infty\} \times \{\Delta\} | \tilde{t}_n, \tilde{x}_n) \right] \\ &= \sum_{n=0}^\infty \tilde{E}_{(v,x)} [I\{\tilde{x}_m \in S, \tilde{t}_m \in [0, \infty), m = 0, 1, \dots, n\} I\{(\tilde{t}_{n+1}, \tilde{x}_{n+1}) = (\infty, \Delta)\}], \end{aligned} \quad (10)$$

where both of the last three equalities are consequences of direct calculations and (9) with the last one further following from the property of conditional expectations. Despite this argument is standard, to improve the readability, we carefully verify the previous equalities for the case of  $v = 0$  as follows.

For the constant function  $x \in S \rightarrow \alpha \in (0, \infty)$ , also written as  $\alpha$ , consider the  $Q$ -function  $q^\alpha$  on  $S_\delta$  as introduced by (7), where  $c$  is replaced with  $\alpha$ . It is known that

$$\alpha \mathbb{E}_x^q \left[ \int_0^\infty e^{-\alpha t} I\{X_t \in S\} dt \right] = \alpha \mathbb{E}_x^{q^\alpha} \left[ \int_0^\infty I\{X_t \in S\} dt \right], \quad \forall x \in S; \quad (11)$$

the formal justification can be found in [11] for a homogeneous Markov pure jump process, and Section 5 of [26] in the more general context of a continuous-time Markov decision process; the intuition is



that the discount factor  $\alpha > 0$  can be interpreted as the killing rate in the model with the extended state space  $S_\delta$ , thus resulting in the  $Q$ -function  $q^\alpha$  on  $S_\delta$ .

Let us rewrite the right hand side of (11) as the expected total cost of a homogeneous discrete-time Markov process as follows; for each  $x \in S$ ,

$$\begin{aligned}
& \alpha \mathbb{E}_x^{q^\alpha} \left[ \int_0^\infty I\{X_t \in S\} dt \right] = \alpha \sum_{n=0}^\infty \mathbb{E}_x^{q^\alpha} \left[ \int_{t_n}^{t_{n+1}} I\{x_n \in S\} dt \right] \\
&= \alpha \sum_{n=0}^\infty \mathbb{E}_x^{q^\alpha} \left[ \mathbb{E}_x^{q^\alpha} \left[ \int_{t_n}^{t_{n+1}} I\{x_n \in S\} dt \middle| \mathcal{F}_{t_n} \right] \right] = \sum_{n=0}^\infty \mathbb{E}_x^{q^\alpha} \left[ I\{x_n \in S\} \int_0^\infty \alpha e^{-\alpha t} e^{-\int_0^t q_{x_n}(s+t_n) ds} dt \right] \\
&= \sum_{n=0}^\infty \mathbb{E}_x^{q^\alpha} \left[ I\{x_n \in S\} \left( 1 - \int_0^\infty e^{-\alpha t - \int_0^t q_{x_n}(t_n+s) ds} q_{x_n}(t_n+t) dt \right) \right] \tag{12}
\end{aligned}$$

where  $\mathcal{F}_{t_n} = \sigma\{(t_m, x_m), 1 \leq m \leq n\}$ , c.f. Theorem 4.13 of Kitaev and Rykov [21], and the last equality is by differentiation by part.

For each fixed  $x \in S$ , one can observe that the process  $(\tilde{t}_n, \tilde{x}_n)$  under  $\tilde{P}_{(0,x)}$  is rather similar to the marked point process  $(t_n, x_n)$  under  $\mathbb{P}_x^{q^\alpha}$ , c.f. (2.9) of [18]. The differences are that the absorbing (cemetery) state for  $x_n$  is now replaced by  $\Delta$ , and we allow the transition from  $[0, \infty) \times S$  to  $\{\infty\} \times \{\Delta\}$  for  $(\tilde{t}_n, \tilde{x}_n)$  with positive probability at each state, whereas  $t_n$  is finite with full probability (given  $x_{n-1} \in S$ ) because of the presence of  $\alpha > 0$  in the  $Q$ -function  $q^\alpha$ . However, due to the presence of  $I\{x_n \in S\}$ , the last expression of (12) can be rewritten as

$$\begin{aligned}
& \sum_{n=0}^\infty \mathbb{E}_x^{q^\alpha} \left[ I\{x_n \in S\} \left( 1 - \int_0^\infty e^{-\alpha t - \int_0^t q_{x_n}(t_n+s) ds} q_{x_n}(t_n+t) dt \right) \right] \\
&= \tilde{E}_{(0,x)} \left[ \sum_{n=0}^\infty I\{\tilde{x}_n \in S\} p^\alpha(\{\infty\} \times \{\Delta\} | \tilde{t}_n, \tilde{x}_n) \right] \\
&= \tilde{E}_{(0,x)} \left[ \sum_{n=0}^\infty I\{\tilde{x}_m \in S, \tilde{t}_m \in [0, \infty), m = 0, 1, \dots, n\} I\{(\tilde{t}_{n+1}, \tilde{x}_{n+1}) = (\infty, \Delta)\} \right]
\end{aligned}$$

for each  $x \in S$ . Thus, (10) is verified for the case of  $v = 0$ . The similar argument applies to the general case which only involves extra notations, and is thus omitted; see [2, 14].

The relation (10) is fundamentally important in this section because it gives two views of the explosion and nonexplosion of the process  $(X_t)$ . Indeed, the right hand side expressions can be viewed as the expected total cost well studied in Markov decision processes, giving rise to the first view, as well as the probability of getting absorbed at  $\{\infty\} \times \{\Delta\}$  of a discrete-time Markov chain well studied in stability, giving the second. The first view easily gives a necessary and sufficient condition for the explosion, whereas the second view easily leads to a sharp drift condition for nonexplosion. We elaborate on this in the rest of this section.

**Theorem 3.1** *Consider the  $Q$ -function  $q$  on  $S$ . Then the (nonhomogeneous) Markov pure jump process  $(X_t)$  with the transition function  $P_q$  is nonexplosive if and only if one of the following two equivalent assertions hold.*

- (a) *There exists some constant  $\alpha \in (0, \infty)$  such that the (homogeneous) discrete-time Markov chain  $(\tilde{t}_n, \tilde{x}_n)$  with the transition probability  $p^\alpha$  given by (9) is Harris recurrent with respect to the irreducibility measure  $\psi$  concentrated on the singleton  $\{\infty\} \times \{\Delta\}$ . (See e.g., [23] for the relevant definitions about the stability of discrete-time Markov chains.)*



(b) For each  $\alpha \in (0, \infty)$ , there is no nontrivial  $[0, 1]$ -valued measurable function  $U$  on  $[0, \infty) \times S$  satisfying

$$U(v, x) = \int_0^\infty \int_S U(v+t, y) \tilde{q}(dy|x, v+t) e^{-\alpha t - \int_0^t q_x(s+v) ds} dt, \quad \forall x \in S, v \in [0, \infty). \quad (13)$$

Here and below, by saying  $U$  is nontrivial, we mean that there exists some  $(v, x)$  such that  $U(v, x) \neq 0$ .

*Proof.* (a) The summand in the last expression of (10) is the first entrance probability of the process  $(\tilde{t}_n, \tilde{x}_n)$ . The statement immediately follows from this observation together with (8), (9) and (10).

(b) Let  $\alpha > 0$  be arbitrarily fixed. The first expression of (10) is the expected total cost for the discrete-time Markov chain  $(\tilde{t}_n, \tilde{x}_n)$  with the cost function given by

$$(v, x) \in ([0, \infty) \times S) \bigcup (\{\infty\} \times \{\Delta\}) \rightarrow I\{x \in S\} \int_0^\infty \alpha e^{-\alpha t} e^{-\int_0^t q_x(s+v) ds} dt;$$

recall the convention adopted in (1). It is well known from the theory of Markov decision processes, see e.g., Propositions 9.8 and 9.10 of Bertsekas and Shreve [3], that the function

$$\begin{aligned} (v, x) \in [0, \infty) \times S &\rightarrow \tilde{E}_{(v, x)} \left[ \sum_{n=0}^\infty I\{\tilde{x}_n \in S\} \int_0^\infty \alpha e^{-\alpha t} e^{-\int_0^t q_{\tilde{x}_n}(s+\tilde{t}_n) ds} dt \right] \\ &= \alpha \int_0^\infty e^{-\alpha t} P_q(v, x, v+t, S) dt \end{aligned}$$

(c.f. (10)) is the minimal solution to the following dynamic programming-type equation

$$\begin{aligned} W(v, x) &= \int_0^\infty \alpha e^{-\alpha t} e^{-\int_0^t q_x(s+v) ds} dt + \int_0^\infty \int_S W(t+v, y) \tilde{q}(dy|x, v+t) e^{-\alpha t - \int_0^t q_x(v+s) ds} dt, \\ &\quad \forall x \in S, v \in [0, \infty), \end{aligned}$$

out of the class of  $[0, 1]$ -valued measurable functions on  $[0, \infty) \times S$ . It is easy to check that a  $[0, 1]$ -valued measurable function  $W$  on  $[0, \infty) \times S$  is a solution to the above equation if and only if  $U = 1 - W$  is a solution to (13). Consequently,

$$(v, x) \in [0, \infty) \times S \rightarrow 1 - \alpha \int_0^\infty e^{-\alpha t} P_q(v, x, v+t, S) dt$$

is the maximal solution to (13) out of the class of  $[0, 1]$ -valued measurable functions on  $[0, \infty) \times S$ . The statement now follows.  $\square$

We come back to part (a) of the previous theorem later. For its part (b), to the best knowledge of ours, it was only stated in the Chinese literature, see e.g., Theorem 7.2 of Chapter 3 in [19], where the continuity of the function  $t \in [0, \infty) \rightarrow q(\Gamma|x, t)$  for each  $x \in S$  and  $\Gamma \in \mathcal{B}(S)$  was assumed, and a different and more complicated argument was followed. The present argument does not require the continuity condition, and is much more transparent.

In practice, it is not easy to apply directly the condition in part (b) of Theorem 3.1 to check for the nonexplosion. Nevertheless, the negation of its equivalent form provides a more applicable condition for the explosion of the process  $(X_t)$ . The statement is as follows.

**Corollary 3.1** *Consider the  $Q$ -function  $q$  on  $S$ . The (nonhomogeneous) Markov pure jump process  $(X_t)$  with the transition function  $P_q$  is explosive if and only if for some  $\alpha > 0$ , there is a nontrivial bounded nonnegative measurable function  $U$  satisfying the following inequality:*

$$U(v, x) \leq \int_0^\infty \int_S U(v+t, y) \tilde{q}(dy|x, v+t) e^{-\alpha t - \int_0^t q_x(s+v) ds} dt, \quad \forall x \in S, v \in [0, \infty). \quad (14)$$

*Proof.* The statement directly follows from the proof of part (b) of Theorem 3.1 and known facts in Markov decision processes. In greater detail, by Proposition 9.10 of Bertsekas and Shreve [3], the function

$$(v, x) \in [0, \infty) \times S \rightarrow \tilde{E}_{(v,x)} \left[ \sum_{n=0}^{\infty} I\{\tilde{x}_n \in S\} \int_0^{\infty} \alpha e^{-\alpha t} e^{-\int_0^t q_{\tilde{x}_n}(s+\tilde{t}_n)ds} dt \right]$$

is the minimal solution to the following inequality

$$\begin{aligned} W(v, x) &\geq \int_0^{\infty} \alpha e^{-\alpha t} e^{-\int_0^t q_x(s+v)ds} dt + \int_0^{\infty} \int_S W(t+v, y) \tilde{q}(dy|x, v+t) e^{-\alpha t - \int_0^t q_x(v+s)ds} dt, \\ &\forall x \in S, v \in [0, \infty) \end{aligned}$$

out of the class of  $[0, 1]$ -valued measurable functions on  $[0, \infty) \times S$ . Now, as in the proof of Theorem 3.1(b), it follows that the function

$$(v, x) \in [0, \infty) \times S \rightarrow 1 - \alpha \int_0^{\infty} e^{-\alpha t} P_q(v, x, v+t, S) dt$$

is the maximal solution to inequality (14) out of the class of  $[0, 1]$ -valued measurable functions on  $[0, \infty) \times S$ . The statement immediately follows. (Note that there is a bounded nonnegative (measurable) solution to inequality (14) if and only if there is a  $[0, 1]$ -valued solution to it.)  $\square$

The condition for explosion of a nonhomogeneous Markov pure jump process in a Borel state space in the above statement seems not stated in the literature. On the other hand, part (a) of Theorem 3.1 naturally leads to a sharp drift condition for the nonexplosion, which is formulated as follows.

**Condition 3.1** *There exists a monotone nondecreasing sequence  $(\tilde{S}_n) \subseteq \mathcal{B}([0, \infty) \times S)$  and a  $[0, \infty)$ -valued measurable function  $V$  on  $[0, \infty) \times S$  such that the following hold.*

(a) *As  $n \uparrow \infty$ ,  $\tilde{S}_n \uparrow [0, \infty) \times S$ .*

(b) *For each  $n = 1, 2, \dots$ ,*

$$\sup_{x \in S_n, t \geq 0} q_x(t) < \infty, \quad (15)$$

*where  $S_n$  denotes the projection of  $\tilde{S}_n$  on  $S$ .*

(c) *As  $n \uparrow \infty$ ,*

$$\inf_{(t,x) \in ([0,\infty) \times S) \setminus \tilde{S}_n} V(t, x) \uparrow \infty. \quad (16)$$

(d) *For some  $\alpha \in (0, \infty)$ ,*

$$\int_0^{\infty} \int_S V(t+v, y) e^{-\alpha t - \int_0^t q_x(s+v)ds} \tilde{q}(dy|x, t+v) dt \leq V(v, x), \quad \forall x \in S, v \in [0, \infty). \quad (17)$$

**Theorem 3.2** *Consider the  $Q$ -function  $q$  on  $S$ . Then the following assertions hold.*

(a) *The (nonhomogeneous) Markov pure jump process  $(X_t)$  with the transition function  $P_q$  is non-explosive if Condition 3.1 is satisfied.*

(b) Suppose  $S$  is a locally compact separable metric space;  $\sup_{x \in \Gamma, t \geq 0} q_x(t) < \infty$  for each compact subset  $\Gamma$  of  $S$ ; and that the transition probability  $p^\alpha$  for the discrete-time Markov chain  $(\tilde{x}_n, \tilde{t}_n)$  is weakly Feller, i.e.,

$$(v, x) \in [0, \infty) \times S \rightarrow \int_S \int_0^\infty f(t, y) p^\alpha(dt \times dy | v, x)$$

is continuous for each bounded continuous function  $f$  on  $[0, \infty) \times S$ . If the (nonhomogeneous) Markov pure jump process  $(X_t)$  is nonexplosive, then Condition 3.1 holds, where one can take  $\tilde{S}_n = T_n \times S_n$  with  $T_n \subseteq [0, \infty)$  and  $S_n \subseteq S$  being compact sets such that

$$\sup_{(t, x) \in T_n \times S_n} V(t, x) < \infty.$$

*Proof.* (a) Suppose Condition 3.1 is satisfied. Let us extend the definition of the function  $V$  to  $([0, \infty) \times S) \cup (\{\infty\} \times \{\Delta\})$  by putting  $V(\infty, \Delta) = 0$ . Firstly, we observe that for each  $n = 0, 1, \dots$ , the set  $\tilde{S}_n \cup (\{\infty\} \times \{\Delta\})$  is petite for the Markov chain  $(\tilde{t}_n, \tilde{x}_n)$  with the transition probability  $p^\alpha$ . Indeed, for each  $(v, x) \in \tilde{S}_n$  for some  $n$ ,

$$\begin{aligned} p^\alpha(\Gamma_1 \times \Gamma_2 | v, x) &\geq \delta_{\{\infty\} \times \{\Delta\}}(\Gamma_1 \times \Gamma_2) \left( 1 - \int_0^\infty e^{-\alpha t - \int_0^t q_x(s+v) ds} q_x(t+v) dt \right) \\ &= \delta_{\{\infty\} \times \{\Delta\}}(\Gamma_1 \times \Gamma_2) \int_0^\infty \alpha e^{-\alpha t} e^{-\int_0^t q_x(s+v) ds} dt \\ &\geq \delta_{\{\infty\} \times \{\Delta\}}(\Gamma_1 \times \Gamma_2) \frac{\alpha}{\alpha + \sup_{x \in S_n, t \geq 0} \{q_x(t)\}}, \end{aligned}$$

c.f. (15). Remember the notation used here that  $S_n$  is the projection of  $\tilde{S}_n$  on  $S$ . The above inequality also trivially holds when  $(v, x) = (\infty, \Delta)$ . By the validity of the previous inequality and (15), the set  $\tilde{S}_n \cup (\{\infty\} \times \{\Delta\})$  is petite for the chain  $(\tilde{t}_n, \tilde{x}_n)$ , see p.121 of Meyn and Tweedie [23]. Next, we note that the function  $V$  is bounded off petite sets, i.e, for each  $m = 0, 1, \dots$ , the set  $\{(v, x) \in ([0, \infty) \times S) \cup (\{\infty\} \times \{\Delta\}) : V(v, x) \leq m\}$  is petite. Indeed, by Condition 3.1, there exists some  $n$  such that

$$\{(v, x) \in ([0, \infty) \times S) \cup (\{\infty\} \times \{\Delta\}) : V(v, x) \leq m\} \subseteq \tilde{S}_n \cup (\{\infty\} \times \{\Delta\}).$$

Since the set of the right hand side is petite as observed earlier, so is the set on the left hand side. The chain  $(\tilde{t}_n, \tilde{x}_n)$  is  $\psi$ -irreducible with the (maximal) irreducibility measure  $\psi$  concentrated on  $\{\infty\} \times \{\Delta\}$ . Now for part (a), one can refer to Theorem 8.4.3 of [23], where we put  $C = \{\infty\} \times \{\delta\}$ , which is clearly petite. (The validity of the proof of Theorem 8.4.3 in [23] does not require the state space of the chain to be locally compact.)

(b) By the local compactness assumption of the Borel space  $S$ , there exist a monotone nondecreasing sequence of open precompact sets  $(T_n^o \times S_n^o) \subseteq \mathcal{B}([0, \infty) \times S)$  such that  $T_n^o \times S_n^o \uparrow [0, \infty) \times S$ . Now the statement is a consequence of the proof of Theorem 9.4.2 in [23]. (There, one can put  $C_0 = A_0 = \{\infty\} \times \{\delta\}$ ,  $A_i = (T_i^o \times S_i^o) \cup (\{\infty\} \times \{\Delta\})$ , and replace  $A_i$  by its closure in (9.37) and (9.38).) It remains to take  $\tilde{S}_n = T_n \times S_n$  as the closure of  $T_n^o \times S_n^o$ .  $\square$

The previous theorem shows that our drift condition for nonexplosion (Condition 3.1) is sharp. To the best of our knowledge, this condition has not been reported in the previous literature for nonhomogeneous Markov pure jump processes.

As an application of Theorem 3.2, we shall apply it to obtaining directly a more applicable sufficient condition for nonexplosion, see Condition 3.2 below, which is related to but at the same time more

general than the one originally obtained by J. Zheng and X. Zheng [35] and recently re-obtained by Chow and Khasminskii [9] for the process in a denumerable state space with a continuous  $Q$ -function. More comments on this are after Corollary 3.2 below.

**Condition 3.2** *There exist a monotone nondecreasing sequence  $(S_n) \subseteq \mathcal{B}(S)$  and a nonnegative measurable function  $V$  on  $[0, \infty) \times S$  such that the following assertions hold.*

(a) *As  $n \uparrow \infty$ ,  $S_n \uparrow S$ .*

(b) *For each  $T > 0$ , as  $n \uparrow \infty$ ,*

$$\inf_{v \in [0, T], x \in S \setminus S_n} V(v, x) \uparrow \infty.$$

(c) *For each  $T > 0$ , and  $n = 1, 2, \dots$ ,*

$$\sup_{v \in [0, T], x \in S_n} q_x(v) < \infty.$$

(d) *For each  $T > 0$ , there is some constant  $\alpha_T \in (0, \infty)$  such that*

$$\int_0^{T-v} \int_S V(t+v, y) e^{-\alpha_T t - \int_0^t q_x(s+v) ds} \tilde{q}(dy|x, t+v) dt \leq V(v, x), \quad \forall v \in [0, T], x \in S.$$

**Corollary 3.2** *Consider a  $Q$ -function  $q$  on  $S$ . The nonhomogeneous Markov pure jump process  $(X_t)$  with the transition function  $P_q$  is nonexplosive if Condition 3.2 is satisfied.*

*Proof.* For each fixed  $T \in [0, \infty)$ , let us consider the following  $Q$ -function on  $S$

$${}_T q(\Gamma|x, t) := q(\Gamma|x, t) I\{t \in [0, T]\}, \quad \forall \Gamma \in \mathcal{B}(S). \quad (18)$$

According to Feller's construction, c.f., (2) and (3), one can see that in particular,

$$P_q(0, x, t, \Gamma) = P_{{}_T q}(0, x, t, \Gamma), \quad \forall x \in S, t \in [0, T], \Gamma \in \mathcal{B}(S). \quad (19)$$

Note that under Condition 3.2, the corresponding version of Condition 3.1 for the  $Q$ -function  ${}_T q$  is satisfied for each  $T \in [0, \infty)$ . Here, keeping in mind the definition (18), only the validity of the corresponding version of (16) needs special explanations, as follows. For each fixed  $T \in [0, \infty)$ , we can modify the definition of  $V$  by putting

$$V_T(v, x) = I\{v > T\} V(0, x) + V(v, x) I\{v \in [0, T]\}$$

for each  $x \in S$ . In this way, we obtain for each  $T \geq 0$ ,

$$\inf_{x \in S \setminus S_n, v \in [0, \infty)} V_T(v, x) = \inf_{x \in S \setminus S_n, v \in [0, T]} V(v, x) \rightarrow \infty, \quad \forall x \in S,$$

as  $n \rightarrow \infty$  under Condition 3.2. Consequently, for each  $T > 0$ , the corresponding version of Condition 3.1 for the  $Q$ -function  ${}_T q$  is satisfied by the function  $V_T$  and the sequence of sets  $\tilde{S}_n = [0, \infty) \times S_n$ . By Theorem 3.2, for each  $T \geq 0$  and  $x \in S$ ,  $P_{{}_T q}(0, x, T, S) = 1$ , and thus

$$P_q(0, x, T, S) = 1, \quad \forall x \in S, T \geq 0,$$

by (19). Now the statement follows from the above equality and Lemma 2.1.  $\square$

The drift or test functions  $V$  appearing in Conditions 3.1 and 3.2, though denoted with the same notation, are not necessarily the same. In our opinion, parts (b) and (c) of Condition 3.2 are easier for practical verifications than parts (b) and (c) of Condition 3.1.

As mentioned earlier, recently, Chow and Khasminskii [9] also studied the nonexplosion of a non-homogeneous Markov pure jump process but in a denumerable state space  $S = \mathbb{Z} = \{0, \pm 1, \pm 2, \dots\}$ ; following a different approach, they obtained a sufficient condition for nonexplosion, which is formulated below in the setup of the present paper for the ease of reference of comparisons, c.f., Theorem 1 of [9].

**Condition 3.3 (Zheng and Zheng, and Chow and Khasminskii)**

(a)  $S = \mathbb{Z}$ , and the function

$$t \in [0, \infty) \rightarrow q(\{j\}|i, t)$$

is continuous for each  $i, j \in S = \mathbb{Z}$ .

(b) There exists a nonnegative function  $V$  on  $[0, \infty) \times \mathbb{Z}$  such that the following assertions hold.

(i)  $v \in [0, \infty) \rightarrow V(v, i)$  is continuously differentiable for each  $i \in \mathbb{Z}$ .

(ii) For each  $T \geq 0$  and  $i \in \mathbb{Z}$ ,

$$\sup_{0 \leq v \leq T} \sum_{j \in \mathbb{Z}} \tilde{q}(\{j\}|i, v) |V(v, j)| < \infty.$$

(iii) For each  $T \geq 0$ , as  $n \uparrow \infty$ ,

$$\inf_{v \in [0, T], |i| > n} V(v, i) \uparrow \infty.$$

(iv) For each  $T \geq 0$ , there is some constant  $\alpha_T \in (0, \infty)$  such that

$$\frac{\partial V(v, i)}{\partial v} + \sum_{j \in \mathbb{Z}} q(\{j\}|i, v) V(v, j) \leq \alpha_T V(v, i), \quad \forall v \in [0, T], i \in \mathbb{Z}. \quad (20)$$

It might be worthwhile to mention the following fact, which we learnt from Professors Mufa Chen (Beijing Normal University) and Floske Spieksma (Leiden University). Condition 3.3 sufficient for the nonexplosion was, to the best of our knowledge, initially obtained by J.Zheng and X.Zheng in the 1980s (also for the case of a denumerable state space and continuous  $Q$ -function like in [9]). However, the authors announced the result without proof in [35], which, in spite of being in English, has not been widely accessible. The corresponding proof, quite different from ours (as the title of that paper suggests), seems only available in the Ph.D thesis of J.Zheng [36] under the supervision of Mufa Chen, which was written in Chinese, and not widely accessible as well.

The following observation shows that Condition 3.3 is stronger than Condition 3.1 and Condition 3.2.

**Proposition 3.1** *If Condition 3.3 is satisfied, then so are Condition 3.1 and Condition 3.2.*

*Proof.* Suppose that Condition 3.3 is satisfied. Since the concerned process is in a denumerable state space, and the  $Q$ -function is continuous, Theorem 3.2 applies, so that the validity of Condition 3.1

follows from the sufficiency of Condition 3.3 for nonexplosion, see Theorem 1 of [9]. To see that Condition 3.2 is satisfied, merely note that for each  $T \geq 0$ ,

$$\begin{aligned}
& \int_0^{T-v} \sum_{j \in \mathbb{Z}} V(t+v, j) e^{-\alpha_T t - \int_0^t q_i(s+v) ds} \tilde{q}(\{j\} | i, t+v) dt \\
&= \int_0^{T-v} \sum_{j \in \mathbb{Z}} V(t+v, j) e^{-\alpha_T t - \int_0^t q_i(s+v) ds} q(\{j\} | i, t+v) dt \\
&\quad + \int_0^{T-v} V(t+v, i) e^{-\alpha_T t - \int_0^t q_i(s+v) ds} q_i(t+v) dt \\
&\leq \int_0^{T-v} e^{-\alpha_T t - \int_0^t q_i(s+v) ds} (\alpha_T + q_i(t+v)) V(t+v, i) dt - \int_0^{T-v} e^{-\alpha_T t - \int_0^t q_i(s+v) ds} \frac{\partial V(v, i)}{\partial v} \Big|_{t+v} dt \\
&\leq V(v, i), \quad \forall v \in [0, T], \quad i \in \mathbb{Z},
\end{aligned}$$

where the first inequality is by (20).  $\square$

It is less transparent how to show directly that Condition 3.3 implies Condition 3.1 without using Theorem 3.2. According to Proposition 3.1, Corollary 3.2 can be viewed as an extension of Theorem 1 of [9] to the case of a general state space and a  $Q$ -function not necessarily continuous, regarding the nonexplosion. However, our arguments are quite different from those in [9] and [36], and are much straightforward in our opinion. Remember, different from [9, 36], we do not require the continuity on the  $Q$ -function here. This is important for applications to continuous-time Markov decision processes [17, 27], where in principle for optimality one may not exclude apriori strategies under which the controlled process does not have a continuous  $Q$ -function. In this connection, we mention that the nonexplosion of nonhomogeneous pure jump processes has been also studied in the context of continuous-time Markov decision processes, where the corresponding version of the following condition, which evidently implies Condition 3.1, has been widely used, see e.g., Assumption A of Guo and Song [18], where the continuity of the drift function was required but can be withdrawn; see [25] and the relevant reference therein. The proof in [18], see the proof of Theorem 3.1 therein, is more complicated and quite different from the one here.

**Condition 3.4** *There exists a monotone nondecreasing sequence  $(S_n) \subseteq \mathcal{B}(S)$  and a  $[0, \infty)$ -valued measurable function  $V$  on  $S$  such that the following hold.*

(a) *As  $n \uparrow \infty$ ,  $S_n \uparrow S$ .*

(b) *For each  $n = 0, 1, \dots$ ,  $\sup_{x \in S_n} V(x) < \infty$  and*

$$\sup_{x \in S_n, v \geq 0} q_x(v) < \infty.$$

(c) *As  $n \uparrow \infty$ ,  $\inf_{x \in S \setminus S_n} V(x) \uparrow \infty$ .*

(d) *For some constant  $\alpha \in (0, \infty)$ ,*

$$\int_S V(y) q(dy | x, v) \leq \alpha V(x), \quad \forall x \in S, v \in [0, \infty).$$

The above condition will be also referred to in the discussions for the homogeneous cases as follows.

To end this section let us say a few words about the homogeneous case, i.e., when the  $Q$ -function  $q(dy | x, s)$  on  $S$  does not depend on time  $s \geq 0$ . Then  $(X_t)$  is a homogeneous Markov pure jump process

with the transition function  $P_q$ . The nonexplosion and explosion for this homogeneous case have been better studied.

In the homogenous case, instead of the discrete-time Markov chain  $(\tilde{t}_n, \tilde{x}_n)$  with the transition probability  $p_\alpha$  given by (9), one can simply focus on the Markov chain  $(\tilde{x}_n)$  whose transition probability can be obtained automatically from  $p_\alpha$ . The reasoning for the nonhomogeneous case in the previous discussions all applies. In particular, we see the following versions of Theorem 3.2, Theorem 3.1 and Corollary 3.1 for the nonexplosion and nonexplosion of homogeneous Markov pure jump process  $(X_t)$ .

**Theorem 3.3** *Consider the homogeneous  $Q$ -function  $q$  on  $S$ . Then the following assertions hold.*

- (a) *The homogeneous Markov pure jump process  $(X_t)$  with the transition function  $P_q$  is nonexplosive if Condition 3.4 is satisfied, where the  $Q$ -function  $q$  does not depend on time  $v$ .*
- (b) *Suppose  $S$  is a locally compact separable metric space, and the functions  $x \in S \rightarrow q_x$  and  $x \in S \rightarrow \int_S f(y)q(dy|x)$  are continuous for each bounded continuous function  $f$  on  $S$ . If the homogeneous Markov pure jump process  $(X_t)$  is nonexplosive, then Condition 3.4 holds, where the sets  $(S_n)$  can be taken as compact sets in  $S$  such that*

$$\sup_{x \in S_n} V(x) < \infty, \quad \forall n = 1, 2, \dots$$

- (c) *For each  $\alpha \in (0, \infty)$ , there is no nontrivial  $[0, 1]$ -valued measurable function  $U$  on  $[0, \infty) \times S$  satisfying*

$$\alpha U(x) = \int_S U(y)q(dy|x), \quad \forall x \in S.$$

- (d) *The process  $(X_t)$  is explosive if and only if there exists a nontrivial bounded nonnegative measurable function  $U$  on  $S$  and some  $\alpha > 0$  such that*

$$\alpha U(x) \leq \int_S U(y)q(dy|x), \quad \forall x \in S.$$

Only part (b) of the above statement is new, whereas the other parts were known in the literature following different methods. In greater detail, the first part of the above theorem is the same as Theorem 2.25 of [6]. Its origin is Theorem (16) in [4], where there was a minor misprint, was obtained by Mufa Chen, using a different method. In the survey [8] and the monograph [7] (see p.167-168 therein) by Mufa Chen, that condition was demonstrated with various examples to be practically applicable. Part (c) of the above theorem coincides with Theorem 2.40 in Chen [6], where this necessary and sufficient condition for nonexplosion is called zero-exit. The origin of the zero-exit condition is in Theorem 7 of [28] obtained by Reuter for the case of a homogeneous continuous-time Markov chain, for which, more discussions are given in [8, 31]. Finally, part (d) of the previous statement was also presented in [6], see Theorem 2.27 therein. The arguments in [6, 28] are quite different from ours and are more complicated in our opinion.

Part (b) of the above theorem was not reported elsewhere. On the other hand, also specialised to the homogeneous case but with a denumerable state space, the second part of Theorem 3.3, asserting the necessity of Condition 3.4 for the nonexplosion was established by Spieksma in [31]. For the case of a homogeneous pure jump process in a general state space, the necessity of Condition 3.4 was mentioned to be unclear in the recent survey [8], see the proof of Theorem 4 therein, as well as the last paragraph of the text in p.223 of [8]. In response to this, we mention that Theorem 3.3(b) provides an affirmative answer when the state space is locally compact separable metrizable, and the  $Q$ -function satisfies some weak Feller-type condition, recall also part (b) of Theorem 3.2 for the nonhomogeneous case. The answer to this question is not yet clear to us in the more general case.



## 4 Applications

The objective of this section is to illustrate possible applications of the results for nonexplosion in the previous section. To be specific yet illuminating, we concentrate on the relations with the stability of homogeneous Markov pure jump processes and continuous-time Markov decision processes with nonstationary strategies, see the discussions after the proofs of Theorem 4.1 and Theorem 4.3.

**Definition 4.1** *Let  $c \in (0, \infty)$  be fixed. A function  $f$  on  $S$  is called a  $c$ -drift function with respect to the  $Q$ -function  $q$  on  $S$  if*

- (a) *it is  $(0, \infty)$ -valued and measurable on  $S$ , and*
- (b) *for each  $x \in S$ ,*

$$\int_S f(y)q(dy|x, s) \leq cf(x) \quad (21)$$

*for all  $s \geq 0$ .*

*(C.f., Assumption A of [18] with  $a$  replaced by  $s$  here.)*

**Remark 4.1** *In (21) of the previous definition, it is implicitly assumed that*

$$\int_S f(y)\tilde{q}(dy|x, s) < \infty$$

*so that the integral*

$$\int_S f(y)q(dy|x, s) := \int_S f(y)\tilde{q}(dy|x, s) - q_x(s)f(x),$$

*i.e., the left hand side of (21), is well-defined and actually finite.*

Throughout this paper, when talking about a  $c$ -drift function, the constant  $c$  is always positive. In particular, each positive constant function is a  $c$ -drift function.

**Definition 4.2** *Let  $f$  be a  $c$ -drift function with respect to the  $Q$ -function  $q$  on  $S$ . A (real-valued) function  $g$  on  $S$  is said to be  $f$ -bounded if  $g$  is measurable on  $S$  and*

$$\|g\|_f := \sup_{x \in S} \frac{|g(x)|}{f(x)} < \infty.$$

The following fact is used frequently below without special reference.

**Lemma 4.1** *Let  $f$  be a  $c$ -drift function with respect to the  $Q$ -function  $q$  on  $S$ . Then for each  $f$ -bounded function  $g$  on  $S$  and for each  $x \in S$  and  $s \geq 0$ , the integral  $\int_S g(y)q(dy|x, s)$  is well defined, and satisfies*

$$\left| \int_S g(y)q(dy|x, s) \right| \leq \|g\|_f f(x)(c + 2q_x(s)).$$

*Proof.* See the appendix. □

**Remark 4.2** For each  $c$ -drift function  $f$  with respect to the  $Q$ -function  $q$  on  $S$ , we automatically extend its definition to  $S_\infty$  by putting

$$f(x_\infty) := 0.$$

The similar extension is done to each  $f$ -bounded function  $g$ .

According to the previous remark, in what follows, we write expressions such as  $\int_S f(y)P_q(dy|0, x, t, dy)$  and  $\mathbb{E}_x^q[f(X_t)]$  interchangeably.

**Lemma 4.2** For a  $c$ -drift function  $f$  with respect to the  $Q$ -function  $q$  on  $S$ ,

$$\int_S f(y)P_q(dy|s, x, t, dy) \leq e^{c(t-s)}f(x), \quad \forall x \in S, \quad s, t \in [0, \infty), \quad s \leq t.$$

Consequently,

$$\int_u^t \mathbb{E}_x^q \left[ \left( \int_S f(y)q(dy|X_s, s) \right)^+ \right] ds < \infty, \quad \forall x \in S, \quad u, t \in [0, \infty), \quad u \leq t.$$

*Proof.* See the appendix. □

The question of interest is when does the following equation holds for a  $c$ -drift function  $f$  on  $S$ ;

$$\mathbb{E}_x^q[f(X_t)] - f(x) = \int_0^t \mathbb{E}_x^q \left[ \int_S f(y)q(dy|X_s, s) \right] ds, \quad \forall x \in S, \quad t \in [0, \infty). \quad (22)$$

Note that by Lemma 4.2, for a  $c$ -drift function  $f$ , the right hand side is well defined, and in fact finite. More precisely, we have the next statement.

**Lemma 4.3** For a  $c$ -drift function  $f$  with respect to the  $Q$ -function  $q$  on  $S$ , relation (22) holds if and only if

$$\mathbb{E}_x^q[f(X_t)] - \mathbb{E}_x^q[f(X_u)] = \int_u^t \mathbb{E}_x^q \left[ \int_S f(y)q(dy|X_s, s) \right] ds, \quad \forall x \in S, \quad u, t \in [0, \infty), \quad u \leq t. \quad (23)$$

Provided that either (22) or (23) holds, we have

$$\int_0^t \mathbb{E}_x^q \left[ \left| \int_S f(y)q(dy|X_s, s) \right| \right] ds < \infty, \quad \forall x \in S, \quad t \in [0, \infty). \quad (24)$$

*Proof.* See the appendix. □

The first assertion of the previous lemma is only needed in the proof of Lemma A.2 in the appendix.

**Theorem 4.1** Let  $f$  be a  $c$ -drift function with respect to the  $Q$ -function  $q$  on  $S$ . Then relation (22) holds if and only if

$$P_{qf}(0, x, t, S_\delta) = 1, \quad \forall x \in S, \quad t \geq 0, \quad (25)$$

i.e., the process  $(X_t)$  with the transition function  $P_{qf}$  is nonexplosive (c.f., Lemma 2.1, and recall that the state  $\delta$  is absorbing for the  $Q$ -function  $q^f$ ).

*Proof.* Let  $x \in S$  be arbitrarily fixed. Then for each  $t \geq 0$ ,

$$\begin{aligned}
& \int_S P_{q^f}(0, x, t, dy) q^f(S|y, t) = \int_S P_{q^f}(0, x, t, dy) \left( \frac{1}{f(y)} \int_S f(z) q(dz|y, t) - c \right) \\
&= \int_S \frac{e^{-ct}}{f(x)} f(y) P_q(0, x, t, dy) \left( \frac{1}{f(y)} \int_S f(z) q(dz|y, t) - c \right) \\
&= \frac{e^{-ct}}{f(x)} \left\{ \int_S \left( \int_S f(z) q(dz|y, t) \right) P_q(0, x, t, dy) - c \int_S f(y) P_q(0, x, t, dy) \right\}, \tag{26}
\end{aligned}$$

where the first equality is by (7), the second equality is by Lemma A.3, and the last equality holds because the second integral inside the bracket therein is finite by Lemma 4.2.

Suppose (22) holds. By Lemma A.2,

$$\begin{aligned}
& \frac{e^{-ct} \mathbb{E}_x^q[f(X_t)]}{f(x)} = 1 + \int_0^t \frac{e^{-cu}}{f(x)} \left( \mathbb{E}_x^q \left[ \int_S f(y) q(dy|X_u, u) \right] - c \mathbb{E}_x^q[f(X_u)] \right) du \\
&= 1 + \int_0^t \left( \int_S P_{q^f}(0, x, u, dy) q^f(S|y, u) \right) du,
\end{aligned}$$

where the second equality is by (26). Applying Lemma A.3 to the left hand side of the last equality, we see

$$P_{q^f}(0, x, t, S) = 1 + \int_0^t \left( \int_S P_{q^f}(0, x, u, dy) q^f(S|y, u) \right) du. \tag{27}$$

On the other hand, by Proposition 2.2 applied to the (conservative and stable)  $Q$ -function  $q^f$ ,

$$\begin{aligned}
& P_{q^f}(0, x, t, \{\delta\}) = \int_0^t \left( \int_S P_{q^f}(0, x, u, dy) q^f(\{\delta\}|y, u) \right) du \\
&= \int_0^t \left( \int_S P_{q^f}(0, x, u, dy) (-q^f(S|y, u)) \right) du \in [0, 1].
\end{aligned}$$

Adding this equality to (27), we see

$$P_{q^f}(0, x, t, S_\delta) = 1, \quad \forall x \in S, \quad t \geq 0.$$

This verifies (25).

Suppose (25) holds. Note that

$$\int_{S_\delta} (1 + I\{y \in S\}) q^f(dy|x, s) = q^f(S|x, s) \leq 0 \leq c(I\{x \in S\} + 1), \quad \forall x \in S_\delta, \quad s \geq 0,$$

i.e.,  $y \rightarrow 1 + I\{y \in S\}$  is a  $c$ -drift function with respect to the  $Q$ -function  $q^f$  on  $S_\delta$ . By (25),

$$P_{q^f}(0, x, t, S_\delta) = 1 = \delta_x(S_\delta) + \int_0^t \left( \int_S q^f(S_\delta|y, u) P_{q^f}(0, x, u, dy) \right) du, \quad \forall t \geq 0, \quad x \in S,$$

where the second equality holds because of the conservativeness of the  $Q$ -function  $q^f$ . By Proposition 2.2 applied to  $\Gamma = \{\delta\}$ , the above equality leads to

$$P_{q^f}(0, x, t, S) = \delta_x(S) + \int_0^t \left( \int_S q^f(S|y, u) P_{q^f}(0, x, u, dy) \right) du, \quad \forall t \geq 0, \quad x \in S.$$

This and (25), which is assumed to be true, imply that (22) holds for the  $c$ -drift function  $y \rightarrow 1 + I\{y \in S\}$  with respect to the  $Q$ -function  $q^f$  on  $S_\delta$ , where  $\mathbb{E}_x^q$  is replaced with  $\mathbb{E}_x^{q^f}$ , and  $S$  is replaced by  $S_\delta$ . By Lemma A.2, after simple calculations we see that for each  $c > 0$ ,

$$e^{ct}\mathbb{E}_x^{q^f}[I\{X_t \in S\}] = I\{x \in S\} + \int_0^t e^{cu} \left( \mathbb{E}_x^{q^f} \left[ \int_S I\{y \in S\} q^f(dy|X_u, u) \right] + c\mathbb{E}_x^{q^f}[I\{X_u \in S\}] \right) du, \\ \forall x \in S, t \geq 0.$$

Applying Lemma A.3 to the left hand side, the above equality reads

$$\begin{aligned} \frac{\mathbb{E}_x^q[f(X_t)]}{f(x)} &= 1 + \int_0^t e^{cu} \left( \mathbb{E}_x^{q^f} \left[ q^f(S|X_u, u) \right] + c\mathbb{E}_x^{q^f}[I\{X_u \in S\}] \right) du, \\ &= 1 + \int_0^t e^{cu} \left\{ \frac{e^{-cu}}{f(x)} \left( \mathbb{E}_x^q \left[ \int_S f(y)q(dy|X_u, u) \right] - c \int_S f(y)P_q(0, x, u, dy) \right) + c\mathbb{E}_x^{q^f}[I\{X_u \in S\}] \right\} du, \\ &= 1 + \int_0^t e^{cu} \left\{ \frac{e^{-cu}}{f(x)} \left( \mathbb{E}_x^q \left[ \int_S f(y)q(dy|X_u, u) \right] - c \int_S f(y)P_q(0, x, u, dy) \right) \right. \\ &\quad \left. + c \frac{e^{-cu}}{f(x)} \int_S f(y)P_q(0, x, u, dy) \right\} du, \\ &= 1 + \int_0^t \frac{\mathbb{E}_x^q \left[ \int_S f(y)q(dy|X_u, u) \right]}{f(x)} du, \quad \forall x \in S, t \geq 0. \end{aligned}$$

where the second equality is by (26), and the third equality is by Lemma A.3. Now (22) follows.  $\square$

Now we are in position to present an application of Theorem 4.1. Meyn and Tweedie [24] considered the nonexplosion and stability of general (continuous-time) Markov processes in locally compact separable metric spaces, where the authors provide drift conditions for both of these two properties. However, the test function therein was assumed apriori to be in the domain of the extended generator of some underlying processes, see the last paragraph of p.522 of [24]. It is not always trivial to verify this fact. Therefore, the statements in [24] cannot be directly applied. Theorem 4.1 and the results obtained in the previous section help decide if a given  $c$ -drift function can be used a test function in the context of [24]; we illustrate this in the homogeneous case but in general Borel spaces, see the next corollary. (Local compactness of the state space was further assumed in [24] because the authors considered general Markov processes, and this additional assumption was needed even to define the notion of nonexplosion therein.)

**Corollary 4.1** *Consider a homogeneous  $Q$ -function  $q$  on the Borel state space  $S$ , i.e.,  $q(dy|x, s) \equiv q(dy|x)$ . A  $c$ -drift function  $f$  is in the domain of the extended generator of the process  $(X_t)$  with the transition function  $P_q$  if there exists a monotone nondecreasing sequence  $(S_n) \subseteq \mathcal{B}(S)$  and a  $[0, \infty)$ -valued measurable function  $V$  on  $S$  such that Condition 3.4(a), (b) and (c) are satisfied, and*

$$\int_S f(y)V(y)q(dy|x) \leq \alpha f(x)V(x), \quad \forall x \in S$$

*for some  $\alpha > 0$ . If in addition,  $S$  is locally compact, and the kernel  $\tilde{q}(dy|x)$  is weakly Feller, then the previous conditions are necessary.*

*Proof.* Following the definition in [24], the  $c$ -drift function  $f$  is in the domain of the extended generator of the process  $(X_t)$  with the transition function  $P_q$  if the relations (24) and (22) hold. Now the statement follows from Theorem 4.1 Lemma 4.3 and Theorem 3.3(a) and (b).  $\square$

The above corollary indicates that, practically, one can concentrate on the  $c$ -drift functions as test functions in order to verify the conditions in [24] for the various stability results for (homogeneous) Markov pure jump processes, see e.g., Conditions (CD0), (CD1) and (CD2) therein.

Next we generalize Theorem 4.1 to a statement, see Theorem 4.3 below, which is of special relevance to the study of continuous-time Markov decision processes with nonstationary strategies. We comment on this after Theorem 4.3 below. To this end, we introduce some additional notations, as follows. For each monotone nondecreasing sequence of subsets  $(S_m) \subseteq \mathcal{B}(S)$ , define the truncated  $Q$ -functions  $q^{(m)}$  on  $S$  by

$$q^{(m)}(\Gamma|x, s) := q(\Gamma|x, s)I\{x \in S_m\}, \quad \forall x \in S, s \geq 0, \Gamma \in \mathcal{B}(S)$$

for each  $m = 0, 1, \dots$ . Clearly, each state  $x \in S \setminus S_m$ , if exists, is absorbing with respect to the  $Q$ -function  $q^{(m)}$ . Without special reference, in what follows, by  $(S_m)$  is always meant such a monotone nondecreasing sequence of subsets  $(S_m) \subseteq \mathcal{B}(S)$  satisfying  $S_m \uparrow S$ . (This is consistent with the notation used in the previous section, too.)

**Theorem 4.2** *Consider the  $Q$ -functions  $q$  and  $(q^{(m)})$  on  $S$ . Let  $x \in S$ ,  $s, t \geq 0$ ,  $s \leq t$  be arbitrarily fixed.*

(a) *For each  $\Gamma \in \mathcal{B}(S_j)$  for some  $j = 0, 1, \dots$ , it holds that*

$$\lim_{m \rightarrow \infty} P_{q^{(m)}}(s, x, t, \Gamma) = P_q(s, x, t, \Gamma).$$

*In fact, for each (and there is always some)  $M = 0, 1, \dots$  such that  $x \in S_M$  and  $\Gamma \in \mathcal{B}(S_M)$ ,*

$$P_{q^{(m)}}(s, x, t, \Gamma) \uparrow P_q(s, x, t, \Gamma)$$

*as  $M \leq m \uparrow \infty$ .*

(b) *For each  $\Gamma \in \mathcal{B}(S)$ ,*

$$\underline{\lim}_{m \rightarrow \infty} P_{q^{(m)}}(s, x, t, \Gamma) \geq P_q(s, x, t, \Gamma).$$

*Proof.* (a) If  $S_m = S$  for some  $m = 0, 1, \dots$ , then the statement holds automatically. Below we consider  $S_m \neq S$  for all  $m = 0, 1, \dots$ . Firstly, note that for each  $m = 0, 1, \dots$ ,

$$P_{q^{(m)}}(s, x, t, \Gamma) = \delta_x(\Gamma) = P_{q^{(m)}}^{(0)}(s, x, t, \Gamma), \quad \forall x \in S \setminus S_m, \Gamma \in \mathcal{B}(S), \quad (28)$$

since  $\sup_{s \geq 0} \{q_x^{(m)}(s)\} = 0$  for each  $x \in S \setminus S_m$ ; recall (2).

Secondly, let us show that for each  $m = 0, 1, \dots$ ,

$$\sum_{i=0}^n P_{q^{(m)}}^{(i)}(s, x, t, \Gamma) \leq \sum_{i=0}^n P_{q^{(m+1)}}^{(i)}(s, x, t, \Gamma), \quad \forall x \in S_m, \Gamma \in \mathcal{B}(S_m), s, t \geq 0, s \leq t, \quad (29)$$

by induction as follows. The case of  $n = 0$  is easily verified. Suppose (29) holds for  $n \leq k$ , and consider when  $n = k + 1$ . For each  $x \in S_m$ ,  $\Gamma \in \mathcal{B}(S_m)$ ,  $s, t \geq 0$ ,  $s \leq t$ ,

$$\begin{aligned}
& \sum_{i=0}^{k+1} P_{q^{(m)}}^{(i)}(s, x, t, \Gamma) = P_{q^{(m)}}^{(0)}(s, x, t, \Gamma) + \sum_{i=1}^{k+1} P_{q^{(m)}}^{(i)}(s, x, t, \Gamma) \\
& \leq P_{q^{(m+1)}}^{(0)}(s, x, t, \Gamma) + \sum_{i=0}^k P_{q^{(m)}}^{(i+1)}(s, x, t, \Gamma) \\
& = P_{q^{(m+1)}}^{(0)}(s, x, t, \Gamma) + \sum_{i=0}^k \int_s^t e^{-\int_s^u q_x^{(m)}(v) dv} \int_S \tilde{q}^{(m)}(dy|x, u) P_{q^{(m)}}^{(i)}(u, y, t, \Gamma) \\
& = P_{q^{(m+1)}}^{(0)}(s, x, t, \Gamma) + \int_s^t e^{-\int_s^u q_x^{(m+1)}(v) dv} \left( \int_{S_m} \tilde{q}^{(m+1)}(dy|x, u) \sum_{i=0}^k P_{q^{(m)}}^{(i)}(u, y, t, \Gamma) \right. \\
& \quad \left. + \int_{S \setminus S_m} \tilde{q}^{(m+1)}(dy|x, u) \sum_{i=0}^k P_{q^{(m)}}^{(i)}(u, y, t, \Gamma) \right) \\
& \leq P_{q^{(m+1)}}^{(0)}(s, x, t, \Gamma) + \int_s^t e^{-\int_s^u q_x^{(m+1)}(v) dv} \int_S \tilde{q}^{(m+1)}(dy|x, u) \sum_{i=0}^k P_{q^{(m+1)}}^{(i)}(u, y, t, \Gamma) \\
& = \sum_{i=0}^{k+1} P_{q^{(m+1)}}^{(i)}(s, x, t, \Gamma)
\end{aligned}$$

where the first inequality is by the inductive supposition, the second equality is by (2), the second inequality is by (28), the definition of the  $Q$ -function  $q^{(m)}$  and the induction supposition, and the last equality is by (2). Thus, (29) holds for the case of  $n = k + 1$ , and by induction, for all  $n = 0, 1, \dots$ .

Thirdly, let us verify that

$$\lim_{m \rightarrow \infty} P_{q^{(m)}}^{(i)}(s, x, t, \Gamma) = P_q^{(i)}(s, x, t, \Gamma), \quad \forall \Gamma \in \mathcal{B}(S), \quad x \in S, \quad s, t \geq 0, \quad s \leq t, \quad (30)$$

for each  $i = 0, 1, \dots$ . Indeed, the case of  $i = 0$  is trivially true. Suppose (30) holds for  $i \leq k$ . Then by (2), the inductive supposition as well as the Lebesgue dominated convergence theorem, we see (30) holds for  $i = k + 1$ , and thus for all  $i = 0, 1, \dots$ .

Now, let us fix some  $x \in S$ , and  $\Gamma \in \mathcal{B}(S_M)$  for some  $M = 0, 1, \dots$ . Without loss of generality, we assume  $x \in S_M$ . Then

$$\begin{aligned}
& \lim_{m \rightarrow \infty} \lim_{n \rightarrow \infty} \sum_{i=0}^n P_{q^{(m)}}^{(i)}(s, x, t, \Gamma) = \lim_{n \rightarrow \infty} \lim_{m \rightarrow \infty} \sum_{i=0}^n P_{q^{(m)}}^{(i)}(s, x, t, \Gamma) \\
& = \lim_{n \rightarrow \infty} \sum_{i=0}^n P_q^{(i)}(s, x, t, \Gamma) = P_q(s, x, t, \Gamma),
\end{aligned}$$

where the first equality holds because  $\sum_{i=0}^n P_{q^{(m)}}^{(i)}(s, x, t, \Gamma)$  increases in  $n$  (resp.  $m \geq M$ ) for each fixed  $m$  (resp.,  $n$ ) by (29), c.f. Theorem A1.6 of [2], the second equality is by (30), and the last equality is by (3). The second assertion of this part of the statement holds because of the previous equality and (3). The first assertion immediately follows from this.

(b) For each  $x \in S$ ,  $s, t \geq 0$ ,  $s \leq t$ , and  $\Gamma \in \mathcal{B}(S)$  and  $m = 0, 1, \dots$ ,

$$\begin{aligned} P_{q^{(m)}}(s, x, t, \Gamma) &= P_{q^{(m)}}\left(s, x, t, \bigcup_{n=0}^{\infty} \left(\Gamma \cap (S_n \setminus S_{n-1})\right)\right) \\ &= \sum_{n=0}^{\infty} P_{q^{(m)}}\left(s, x, t, \Gamma \cap (S_n \setminus S_{n-1})\right), \end{aligned}$$

where  $S_0 \setminus S_{-1} := S_0$ ; recall that  $(S_n)$  is a monotone nondecreasing sequence. By the Fatou lemma,

$$\liminf_{m \rightarrow \infty} P_{q^{(m)}}(s, x, t, \Gamma) \geq \sum_{n=0}^{\infty} \liminf_{m \rightarrow \infty} P_{q^{(m)}}\left(s, x, t, \Gamma \cap (S_n \setminus S_{n-1})\right) = P_q(s, x, t, \Gamma),$$

where the last equality is by part (a) of the statement.  $\square$

**Condition 4.1** *There exists a monotone nondecreasing sequence of subset sets  $(S_m)_{m=0}^{\infty} \subseteq \mathcal{B}(S)$  such that*

$$S_m \uparrow S, \quad (31)$$

and

$$\sup_{s \geq 0, x \in S_m} \{q_x(s)\} < \infty, \quad \forall m = 0, 1, \dots$$

The above condition is practically easily verifiable; it is automatically satisfied if  $x \in S \rightarrow \sup_{s \geq 0} q_x(s)$  is majorized by some real-valued measurable function. Therefore, this condition is satisfied by the induced (nonhomogeneous)  $Q$ -function under each strategy considered in e.g., [15, 16], see Assumption A therein.

**Theorem 4.3** *Suppose Condition 4.1 is satisfied. Let  $f$  be a  $c$ -drift function for the  $Q$ -function  $q$  on  $S$ . Then for*

$$\begin{aligned} \mathbb{E}_x^q[g(X_t)] - g(x) &= \int_0^t \mathbb{E}_x^q \left[ \left( \int_S g(z) q(dz | X_s, s) \right) \right] ds \\ &= \mathbb{E}_x^q \left[ \int_0^t \left( \int_S g(z) q(dz | X_s, s) \right) ds \right], \quad \forall x \in S, t \geq 0 \end{aligned} \quad (32)$$

to hold for each  $f$ -bounded function  $g$  on  $S$  satisfying

$$\int_0^t \mathbb{E}_x^q \left[ \left| \int_S g(z) q(dz | X_t, t) \right| \right] dt < \infty, \quad \forall x \in S, t \geq 0, \quad (33)$$

it is necessary and sufficient that one of the following equivalent (by Theorem 4.1) conditions hold:

- (a) The relation (22) holds.
- (b) The process  $(X_t)$  with the transition function  $P_{q^f}$  is nonexplosive.

*Proof.* Throughout this proof, let  $x \in S$  and  $t \geq 0$  be arbitrarily fixed. Consider the sequence of sets  $(S_m)$  in Condition 4.1 and the associated  $Q$ -functions  $(q_m)$ . Then, one can check that for each



$m = 0, 1, \dots$ , the function  $f$  is a  $c$ -drift function on  $S$  for the  $Q$ -function  $q_m$ . As a result, for each  $m = 0, 1, \dots$ ,

$$\mathbb{E}_x^{q^{(m)}}[g(X_t)] - g(x) = \int_0^t \mathbb{E}_x^{q^{(m)}} \left[ \left( \int_S g(z) q^{(m)}(dz|X_s, s) \right) \right] ds \quad (34)$$

for each  $f$ -bounded function  $g$  by Lemma A.1; recall that  $\sup_{x \in S, s \geq 0} \{q_x^{(m)}(s)\} < \infty$ .

The necessity part of the statement is clear because of Theorem 4.1 (see also Lemma 4.3). Therefore, we only prove the sufficiency part as follows. Consider the function  $g$  that is  $f$ -bounded and satisfies (33). By (33), for almost all  $s \in [0, t]$  (with respect to the Lebesgue measure),

$$\mathbb{E}_x^q \left[ \left| \int_S g(z) q(dz|X_s, s) \right| \right] < \infty.$$

Lemma A.4 implies that

$$\lim_{m \rightarrow \infty} \mathbb{E}_x^{q^{(m)}} \left[ \int_S g(y) q^{(m)}(dy|X_s, s) \right] = \mathbb{E}_x^q \left[ \int_S g(y) q(dy|X_s, s) \right]$$

for almost all  $s \in [0, t]$ . By (45) with  $t$  being replaced by  $s \in [0, t]$ , which holds for all  $x \in S$  (as mentioned immediately after it), and (33), one can apply the Lebesgue dominated convergence theorem for that

$$\lim_{m \rightarrow \infty} \int_0^t \mathbb{E}_x^{q^{(m)}} \left[ \left( \int_S g(z) q^{(m)}(dz|X_s, s) \right) \right] ds = \int_0^t \mathbb{E}_x^q \left[ \left( \int_S g(z) q(dz|X_s, s) \right) \right] ds. \quad (35)$$

Note that the above relation holds if  $g$  is replaced by  $f$ , because (22) is supposed to hold, and Lemma 4.3, see (24) therein.

Next, let us show that

$$\lim_{m \rightarrow \infty} \mathbb{E}_x^{q^{(m)}}[g(X_t)] = \mathbb{E}_x^q[g(X_t)] \quad (36)$$

as follows. By Lemma A.3 applied to the  $Q$ -function  $q^{(m)f}$ ,

$$\frac{e^{-ct}}{f(x)} \int_{\Gamma} f(y) P_{q^{(m)}}(0, x, t, dy) = P_{q^{(m)f}}(0, x, t, \Gamma), \quad \forall \Gamma \in \mathcal{B}(S). \quad (37)$$

On the other hand,

$$\begin{aligned} & \overline{\lim}_{m \rightarrow \infty} \int_{\Gamma} f(y) P_{q^{(m)}}(0, x, t, dy) = \overline{\lim}_{m \rightarrow \infty} \int_S f(y) I\{y \in \Gamma\} P_{q^{(m)}}(0, x, t, dy) \\ & \leq \overline{\lim}_{m \rightarrow \infty} \int_S f(y) P_{q^{(m)}}(0, x, t, dy) \\ & = f(x) + \overline{\lim}_{m \rightarrow \infty} \int_0^t \int_S \left( \int_S f(z) q^{(m)}(dz|y, s) \right) P_{q^{(m)}}(0, x, s, dy) ds \\ & = f(x) + \int_0^t \int_S \left( \int_S f(z) q(dz|y, s) \right) P_q(0, x, s, dy) ds \\ & = \mathbb{E}_x^q[f(X_t)] \leq e^{ct}(x) < \infty, \quad \forall \Gamma \in \mathcal{B}(S), \end{aligned}$$

where the second equality holds because (34) holds for  $g = f$ , and the third equality holds due to that (35) is valid for  $g = f$ , see the remark immediately after (35), the forth equality is by validity of (22)

for the function  $f$ , and the second to the last inequality is by Lemma 4.2. Therefore, keeping in mind Theorem 4.2(b) applied to the  $Q$ -function  $q^{(m)}$ , one can refer to Proposition A.1 for that

$$\lim_{m \rightarrow \infty} \int_{\Gamma} f(y) P_{q^{(m)}}(0, x, t, dy) = \int_{\Gamma} f(y) P_q(0, x, t, dy), \quad \forall \Gamma \in \mathcal{B}(S).$$

The above equality and (37) imply

$$\lim_{m \rightarrow \infty} P_{q^{(m)}f}(0, x, t, \Gamma) = \frac{e^{-ct}}{f(x)} \int_{\Gamma} f(y) P_q(0, x, t, dy) = P_{qf}(0, x, t, \Gamma), \quad \forall \Gamma \in \mathcal{B}(S), \quad (38)$$

where the second equality is by Lemma A.3. It follows from the previous setwise convergence that

$$\begin{aligned} & \lim_{m \rightarrow \infty} \frac{e^{-ct}}{f(x)} \int_S g(y) P_{q^{(m)}}(0, x, t, dy) = \lim_{m \rightarrow \infty} \frac{e^{-ct}}{f(x)} \int_S \frac{g(y)}{f(y)} f(y) P_{q^{(m)}}(0, x, t, dy) \\ &= \lim_{m \rightarrow \infty} \int_S P_{q^{(m)}f}(0, x, t, dz) \frac{g(y)}{f(y)} = \frac{e^{-ct}}{f(x)} \int_S \frac{g(y)}{f(y)} f(y) P_q(0, x, t, dy) \\ &= \frac{e^{-ct}}{f(x)} \int_S g(y) P_q(0, x, t, dy), \end{aligned}$$

where the second equality is by (37), and for the third equality we recall (38) and that the measurable function  $y \in S \rightarrow \frac{g(y)}{f(y)}$  is bounded on  $S$ . Consequently, (36) holds.

Now the statement follows from (35), (36) and (34).  $\square$

The previous theorem is closely related to the study of continuous-time Markov decision processes, where each Markov strategy induces a nonhomogeneous Markov pure jump processes. For a concrete example, we mention that in the proof of Theorem 4.1 of Guo and Rieder [15], where the existence of a  $c$ -drift function  $w = f$  (in their notation) was assumed (under each Markov strategy), the authors' arguments precisely relied on the validity of the formula (32) for each  $w$ -bounded function, for which the authors imposed further conditions. Theorem 4.3 clarified which conditions are really needed. In the homogeneous denumerable case, the simplified version of Theorem 4.3 was obtained in [30]. However, in [15] to conclude the optimality of a strategy out of the class of Markov strategies rather than only the class of stationary strategies, one does need the present nonhomogeneous and general state space version of the theorem. On the other hand, here we extend the arguments and results in Spieksma [30] from the homogeneous denumerable case to the nonhomogeneous general state space case with some additional preparations; e.g., now we had to justify Theorem 4.2, whereas for the homogeneous denumerable case Proposition 2.14 of Anderson [1] was directly referred to in [31]. In that connection, we mention that although the relevant part to [30] in Proposition 2.14 of [1] is correct, the general conclusion of Proposition 2.14 of [1] does not hold; see [5].

## 5 Conclusion

In this paper, we obtained new conditions for nonexplosion and explosion for nonhomogeneous Markov pure jump processes in Borel state spaces. The conditions are practical and in the meanwhile sharp; e.g., the one for nonexplosion is necessary if the state space is locally compact and the  $Q$ -function satisfies weak Feller-type conditions. For the homogeneous case, some known conditions are naturally retrieved. The argument here extends the one in [30, 31] which deal with the (homogeneous) continuous-time Markov chains, and is quite transparent and different from those in [9, 36] for nonhomogeneous denumerable state space case, and [4, 6] for the homogeneous general state space case. In particular, this argument easily shows the sharpness of the obtained condition, which partially

addresses the remark made by Mufa Chen in the recent survey [8] about the necessity of his condition for nonexplosion in the homogeneous case in an uncountable state space, see the proof of Theorem 4 as well as the last paragraph of the text in p.223 of [8]. The relations between nonexplosion and the studies of the stability of homogeneous Markov pure jump processes and continuous-time Markov decision processes were demonstrated.

## A Appendix

In this appendix, we present some auxiliary results and the proofs of some lemmas stated in the previous sections.

**Proposition A.1** *Let  $\mu_n$  and  $\mu$  be  $[0, \infty]$ -valued measures on the Borel space  $X$ , and  $f_n$  (resp.,  $f$ ) be  $(-\infty, \infty)$ -valued  $\mu_n$ -integrable (resp., measurable) functions thereon, and  $g_n$  be nonnegative measurable functions on  $X$ . Suppose  $\varliminf_{n \rightarrow \infty} \mu_n(\Gamma) \geq \mu(\Gamma)$  for each  $\Gamma \in \mathcal{B}(X)$ , and  $f_n(x) \rightarrow f(x)$  for each  $x \in X$ . If  $|f_n(x)| \leq g_n(x)$  for each  $x \in X$  and  $n \geq 1$ , and*

$$\varlimsup_{n \rightarrow \infty} \int_X g_n(y) \mu_n(dy) \leq \int_X \varliminf_{n \rightarrow \infty} g_n(y) \mu(dy) < \infty,$$

*then  $f$  is  $\mu$ -integrable, and*

$$\lim_{n \rightarrow \infty} \int_X f_n(y) \mu_n(dy) = \int_X f(y) \mu(dy) \in (-\infty, \infty).$$

*Proof.* See Theorem 2.4 of [29] □

*Proof of Lemma 2.1.* Let us firstly justify that (b), (c) and (d) are equivalent. Clearly, (d) implies (b), which in turn implies (c). That (c) implies (b) is true because clearly  $t \rightarrow P(0, x, t, S)$  is monotone nonincreasing in  $t \geq 0$ . Now suppose (b) is true. By the Chapman-Kolmogorov equation, we see that for each  $x \in S$  and  $0 \leq s \leq t$ ,

$$P_q(s, y, t, S) = 1 \tag{39}$$

for almost all  $y$  with respect to  $P_q(0, x, s, dy)$ . Now suppose for contradiction that (d) does not hold, so that there exist some  $0 \leq s < t$  and  $x \in S$  such that

$$P_q(s, x, t, S) < 1. \tag{40}$$

Then

$$\begin{aligned} 1 &= P_q(0, x, t, S) = P_q(0, x, s, \{x\})P(s, x, t, S) + \int_{S \setminus \{x\}} P_q(0, x, s, dy)P_q(s, y, t, S) \\ &= P_q(0, x, s, \{x\})P(s, x, t, S) + P_q(0, x, s, S \setminus \{x\}) \\ &< P_q(0, x, s, \{x\}) + P_q(0, x, s, S \setminus \{x\}) = 1, \end{aligned}$$

where the first equality and the last equality are by assumption that (b) holds, the second equality is by the Chapman-Kolmogorov equation, the third equality is by (39), and the inequality is by (40). This is a desired contradiction, and thus assertions (b), (c) and (d) are equivalent.

It is clear that (a) implies (b), c.f. Proposition 2.3 and (6). That (b) implies (a) follows from the relation  $\{t_\infty = \infty\} = \bigcap_{n=1}^\infty \{X_n \in S\}$ , c.f. (6). The proof is completed. □

*Proof of Lemma 4.1.* It is clear that  $\int_S g(y)q(dy|x, s)$  is well defined. For the second part of the statement, note that for each  $x \in S$  and  $s \geq 0$ ,

$$\begin{aligned} & \int_S g(y)\tilde{q}(dy|x, s) - g(x)q_x(s) \geq -\|g\|_f \int_S f(y)\tilde{q}(dy|x, s) - \|g\|_f f(x)q_x(s) \\ &= -\|g\|_f \int_S f(y)q(dy|x, s) - 2\|g\|_f f(x)q_x(s) \geq -\|g\|_f cf(x) - 2\|g\|_f f(x)q_x(s) \\ &= -\|g\|_f f(x)(c + 2q_x(s)), \end{aligned}$$

where the second inequality is by the fact that  $f$  is a  $c$ -drift function. Thus,

$$\int_S g(y)q(dy|x, s) \geq -\|g\|_f f(x)(c + 2q_x(s)),$$

the integral on the left hand side being well defined. Similarly, one can check that

$$\int_S g(y)q(dy|x, s) \leq \|g\|_f f(x)(c + 2q_x(s)),$$

as required.  $\square$

*Proof of Lemma 4.2.* The first assertion is by Theorem 3.1 of [16], where the continuity of the  $Q$ -function can be withdrawn. The second assertion follows from the first assertion because for each  $x \in S$  such that  $\int_S f(y)q(dy|x, s) \geq 0$ ,

$$\left( \int_S f(y)q(dy|x, s) \right)^+ = \int_S f(y)q(dy|x, s) \leq cf(x), \quad \forall s \geq 0,$$

whereas for each  $x \in S$  such that  $\int_S f(y)q(dy|x, s) < 0$ ,

$$\left( \int_S f(y)q(dy|x, s) \right)^+ = 0 \leq cf(x), \quad \forall s \geq 0.$$

The statement is proved.  $\square$

**Lemma A.1** *Let  $f$  be a  $c$ -drift function with respect to the  $Q$ -function  $q$  on  $S$ . Suppose*

$$\sup_{x \in S, s \geq 0} \{q_x(s)\} < \infty.$$

*Then for each  $x \in S$ ,  $0 \leq s \leq t$ , and  $f$ -bounded (measurable) function  $g$  on  $S$ ,*

$$\int_S g(z)P_q(s, x, t, dz) = g(x) + \int_s^t \left( \int_S g(z)q(dz|y, u) \right) P_q(s, x, u, dy)du.$$

*In particular,*

$$P_q(s, x, t, S) = 1, \quad \forall x \in S, s, t \in [0, \infty), s \leq t. \quad (41)$$

*Proof.* The first assertion immediately follows from Proposition 2.2 and Lemma 4.2. The last assertion follows because  $f \equiv 1$  is a  $c$ -drift function, and one can put  $g \equiv 1$  in the first assertion.  $\square$

*Proof of Lemma 4.3.* We only need justify that (22) implies (23) as follows. Let some  $x \in S$  and  $t \in [0, \infty)$  be arbitrarily fixed. Since (22) holds by assumption,

$$\int_0^t \mathbb{E}_x^q \left[ \left( \int_S f(y)q(dy|X_s, s) \right)^- \right] ds = \int_0^t \mathbb{E}_x^q \left[ \left( \int_S f(y)q(dy|X_s, s) \right)^+ \right] ds - \mathbb{E}_x^q[f(X_t)] + f(x) < \infty$$

by Lemma 4.2. Consequently, (24) holds. According to the Fubini theorem,

$$\int_0^t \mathbb{E}_x^q \left[ \int_S f(y) q(dy|X_s, s) \right] ds = \mathbb{E}_x^q \left[ \int_0^t \left( \int_S f(y) q(dy|X_s, s) \right) ds \right] \in (-\infty, \infty).$$

Now for each  $0 \leq u \leq t$ , (22) implies

$$\begin{aligned} \mathbb{E}_x^q[f(X_t)] - f(x) &= \mathbb{E}_x^q \left[ \int_0^t \int_S f(y) q(dy|X_s, s) ds \right] \\ &= \mathbb{E}_x^q \left[ \int_0^u \int_S f(y) q(dy|X_s, s) ds \right] + \mathbb{E}_x^q \left[ \int_u^t \int_S f(y) q(dy|X_s, s) ds \right] \\ &= \mathbb{E}_x^q[f(X_u)] - f(x) + \int_u^t \mathbb{E}_x^q \left[ \int_S f(y) q(dy|X_s, s) \right] ds. \end{aligned}$$

All the expressions are finite. Now (23) follows.  $\square$

**Lemma A.2** *For a  $c$ -drift function  $f$  with respect to the  $Q$ -function on  $S$  that satisfies relation (22), it holds that for each  $d \in (-\infty, \infty)$ ,  $x \in S$  and  $t \geq 0$ ,*

$$e^{dt} \mathbb{E}_x^q[f(X_t)] = f(x) + \int_0^t e^{du} \left( \mathbb{E}_x^q \left[ \int_S f(y) q(dy|X_u, u) \right] + d \mathbb{E}_x^q[f(X_u)] \right) du.$$

*Proof.* Keeping in mind Lemmas 4.2 and 4.3 (and especially (24)), direct calculations give

$$\begin{aligned} & \int_0^t e^{ds} \mathbb{E}_x^q \left[ \int_S f(y) q(dy|X_s, s) \right] ds \\ &= \int_0^t \int_0^s \left( de^{du} du \right) \mathbb{E}_x^q \left[ \int_S f(y) q(dy|X_s, s) \right] ds + \mathbb{E}_x^q[f(X_t)] - f(x) \\ &= \int_0^t de^{du} \left( \int_u^t \mathbb{E}_x^q \left[ \int_S f(y) q(dy|X_s, s) \right] ds \right) du + \mathbb{E}_x^q[f(X_t)] - f(x) \\ &= e^{dt} \mathbb{E}_x^q[f(X_t)] (e^{dt} - 1) - \int_0^t de^{du} \mathbb{E}_x^q[f(X_u)] du + \mathbb{E}_x^q[f(X_t)] - f(x), \end{aligned}$$

where the first and the third equalities are by (23). All the expressions are finite. The statement now follows.  $\square$

**Lemma A.3** *Consider a  $c$ -drift function  $f$  with respect to the  $Q$ -function  $q$  on  $S$ . For each  $x \in S$ ,  $s, t \geq 0$ ,  $s \leq t$  and  $\Gamma \in \mathcal{B}(S)$ , the following relation holds;*

$$P_{qf}(s, x, t, \Gamma) = \frac{e^{-c(t-s)}}{f(x)} \int_{\Gamma} f(y) P_q(s, x, t, dy).$$

*Proof.* By Feller's construction, c.f., (2) and (3), it suffices to show

$$\sum_{m=0}^n P_{qf}^{(m)}(s, x, t, \Gamma) = \frac{e^{-c(t-s)}}{f(x)} \int_{\Gamma} f(y) \left( \sum_{m=0}^n P_q^{(m)}(s, x, t, dy) \right), \quad x \in S, \quad s, t \geq 0, \quad s \leq t, \quad \Gamma \in \mathcal{B}(S), \quad (42)$$

as follows. Let  $x \in S$ ,  $s, t \geq 0$ ,  $s \leq t$ ,  $\Gamma \in \mathcal{B}(S)$  be arbitrarily fixed. Consider  $n = 0$ . Then

$$\begin{aligned} & \frac{e^{-c(t-s)}}{f(x)} \int_{\Gamma} f(y) P_q^{(0)}(s, x, t, dy) = \frac{e^{-c(t-s)}}{f(x)} \int_{\Gamma} f(y) e^{-\int_s^t q_x(y) dv} \delta_x(dy) = \delta_x(\Gamma) e^{-\int_s^t q_x(y) dv - c(t-s)} \\ &= P_{qf}^{(0)}(s, x, t, \Gamma). \end{aligned} \quad (43)$$

Now assume that (42) holds for  $n = k$ . Then

$$\begin{aligned}
& \frac{e^{-c(t-s)}}{f(x)} \int_{\Gamma} f(y) \left( \sum_{m=0}^{k+1} P_q^{(m)}(s, x, t, dy) \right) \\
&= P_{q^f}^{(0)}(s, x, t, \Gamma) + \frac{e^{-c(t-s)}}{f(x)} \int_{\Gamma} f(y) \left( \sum_{m=1}^{k+1} P_q^{(m)}(s, x, t, dy) \right) \\
&= P_{q^f}^{(0)}(s, x, t, \Gamma) + \frac{e^{-c(t-s)}}{f(x)} \int_{\Gamma} f(y) \left( \sum_{m=0}^k P_q^{(m+1)}(s, x, t, dy) \right) \\
&= P_{q^f}^{(0)}(s, x, t, \Gamma) + \frac{e^{-c(t-s)}}{f(x)} \int_{\Gamma} f(y) \left( \sum_{m=0}^k \int_s^t e^{-\int_s^u q_x(v) dv} \left( \int_S \tilde{q}(dz|x, u) P_q^{(m)}(u, z, t, dy) \right) du \right) \\
&= P_{q^f}^{(0)}(s, x, t, \Gamma) + \int_s^t e^{-\int_s^u (q_x(v)-c) dv} \int_S \frac{f(z)}{f(x)} \tilde{q}(dz|x, u) \left( \int_{\Gamma} \frac{f(y)}{f(z)} e^{-c(t-u)} \sum_{m=0}^k P_q^{(m)}(u, z, t, dy) \right) du \\
&= P_{q^f}^{(0)}(s, x, t, \Gamma) + \sum_{m=0}^k P_{q^f}^{(m+1)}(s, x, t, \Gamma),
\end{aligned}$$

where the first equality is by (43), the third equality is by (2), and the last equality is by (2), the inductive supposition and (7). Now (42) holds for  $n = k+1$ , and thus for all  $n = 0, 1, \dots$  by induction.  $\square$

**Lemma A.4** Consider the  $Q$ -function  $q$  on  $S$ . Let some  $x \in S$  and  $t \geq 0$  be arbitrarily fixed. Suppose  $g$  is a measurable function on  $S$  such that the following integral is well defined;

$$\int_S g(y) q(dy|x, t) \in (-\infty, \infty),$$

and

$$\mathbb{E}_x^q \left[ \left| \int_S g(y) q(dy|X_t, t) \right| \right] < \infty. \quad (44)$$

Then

$$\lim_{m \rightarrow \infty} \mathbb{E}_x^{q^{(m)}} \left[ \int_S g(y) q^{(m)}(dy|X_t, t) \right] = \mathbb{E}_x^q \left[ \int_S g(y) q(dy|X_t, t) \right].$$

*Proof.* Let  $m = 0, 1, \dots$  be fixed. Then

$$\begin{aligned}
& \mathbb{E}_x^{q^{(m)}} \left[ \left| \int_S g(y) q^{(m)}(dy|X_t, t) \right| \right] = \mathbb{E}_x^{q^{(m)}} \left[ \left| \int_S g(y) q(dy|X_t, t) I\{X_t \in S_m\} \right| \right] \\
&= \int_{S_m} P_{q^{(m)}}(0, x, t, dz) \left| \int_S g(y) q(dy|z, t) \right| \leq \int_{S_m} P_q(0, x, t, dz) \left| \int_S g(y) q(dy|z, t) \right| \\
&\leq \int_S P_q(x, 0, t, dz) \left| \int_S g(y) q(dy|z, t) \right| \\
&< \infty, \quad \forall x \in S_m,
\end{aligned} \quad (45)$$

where the first inequality is by Theorem 4.2 (c.f., (29) and Feller's construction), and the last inequality is by (44). The above relation trivially holds for  $x \in S \setminus S_m$ , too. This fact and Theorem 4.2(b) validate Proposition A.1; recall that

$$\lim_{m \rightarrow \infty} \int_S g(y) q^{(m)}(dy|x, t) = \int_S g(y) q(dy|x, t)$$

for each  $x \in S$ . It follows that the statement of this lemma holds.  $\square$

## References

- [1] Anderson, W. (1991). *Continuous-time Markov Chains*. Springer, New York.
- [2] Bäuerle, N. and Rieder, U. (2011). *Markov Decision Processes with Applications to Finance*. Springer, Berlin.
- [3] Bertsekas, D. and Shreve, S. (1978). *Stochastic Optimal Control*. Academic Press, New York.
- [4] Chen, M. (1986). Coupling for jump processes. *Acta Math. Sin. (English Series)*, **2**, 123-136.
- [5] Chen, M. (1996). A comment on the book “Continuous-time Markov Chains” by W. J. Anderson. *Chinese J. Appl. Probab. Stat.*, **12**, 55-59. Updated version also available on arXiv:1412.5856.
- [6] Chen, M. (2004). *From Markov Chains to Non-equilibrium Particle Systems*. World Scientific, Singapore.
- [7] Chen, M. (2005). *Eigenvalues, Inequalities, and Ergodic Theory*. Springer, London.
- [8] Chen, M. (2015). Practical criterion for uniqueness of q-processes. *Chinese J. Appl. Probab. Stat.*, **31**, 213-224. Also available on arXiv:1503.02119.
- [9] Chow, P. and Khasminskii, R. (2011). Methods of Lyapunov functions for analysis of absorption and explosion in Markov chains. *Problems of Information Transmission*, **47**, 232-250. (Translation of the original Russian text at *Problemy Peredachi Informatsii* **47**, (2011), 19-38.)
- [10] Eibeck, A. and Wagner, W. (2003) Stochastic interacting particle systems and nonlinear kinetic equations. *Ann. Appl. Probab.*, **13**, 845-889.
- [11] Feinberg, E. and Fei, J. (2009). An inequality for variances of the discounted rewards. *J. Appl. Probab.*, **46**, 1209-1212.
- [12] Feinberg, E., Mandava, M. and Shiryaev, A. (2014). On solutions of Kolmogorov’s equations for nonhomogeneous jump Markov processes. *J. Math. Anal. Appl.*, **411**, 261-270.
- [13] Feller, W. (1940). On the integro-differential equations of purely discontinuous Markoff processes. *Trans. Amer. Math. Soc.*, **48**, 488-515.
- [14] Forwick, L., Schäl, M. and Schmitz, M. (2004). Piecewise deterministic Markov control processes with feedback controls and unbounded costs. *Acta Appl. Math.*, **82**, 239-267.
- [15] Guo, X., Rieder, U. 2006. Average optimality for continuous-time Markov decision processes in Polish spaces. *Ann. Appl. Probab.*, **16**, 730-756.
- [16] Guo, X. (2007). Continuous-time Markov decision processes with discounted rewards: the case of Polish spaces. *Math. Oper. Res.*, **32**, 73-87.
- [17] Guo, X., Hernández-Lerma, O. (2009). *Continuous-Time Markov Decision Processes: Theory and Applications*. Springer, Heidelberg.
- [18] Guo, X. and Song, X. (2011). Discounted continuous-time constrained Markov decision processes in Polish spaces. *Ann. Appl. Probab.*, **21**, 2016-2049.
- [19] Hu, D. (2013). *Analytic Theory of Markov Processes in General State Spaces* (in Chinese). Wuhan University Press, Wuhan. Republication of the edition originally published in 1985.



- [20] Jacod, J. (1975). Multivariate point processes: predictable projection, Radon-Nykodym derivatives, representation of martingales. *Z. Wahrscheinlichkeitstheorie verw. Gebiete.*, **31**, 235-253.
- [21] Kitaev, M. and Rykov, V. (1995). *Controlled Queueing Systems*. CRC Press, Boca Raton.
- [22] Menshikov, M. and Petritis, D. (2014). Explosion, implosion, and moments of passage times for continuous-time Markov chains: a semimartingale approach. *Stoch. Proc. Appl.*, **124**, 2388-2414.
- [23] Meyn, S. and Tweedie, R. (1993). *Markov Chains and Stochastic Stability*. Springer, London.
- [24] Meyn, S. and Tweedie, R. (1993). Stability of Markov processes III: Forster-Lyapunov criteria for continous time processes. *Adv. Appl. Probab.*, **25**, 518-548.
- [25] Piunovskiy, A., Zhang, Y. (2011). Discounted continuous-time Markov decision processes with unbounded rates: the convex analytic approach. *SIAM J. Control Optim.*, **49**, 2032-2061.
- [26] Piunovskiy, A. (2015). Randomized and relaxed strategies in continuous-time Markov decision processes. *SIAM J. Control Optim.*, to appear.
- [27] Prieto-Rumeau, T., Hernández-Lerma, O. (2012). *Selected Topics in Continuous-Time Controlled Markov Chains and Markov Games*. Imperial College Press, London.
- [28] Reuter, G. (1957). Denumerable Markov processes and the associated contraction semigroups on  $l$ . *Acta Math.*, **97**, 1-46.
- [29] Serfozo, R. (1982). Convergence of Lebesgue integrals with varying measures. *Sankhya*, **44**, 380-402.
- [30] Spieksma, F. (2012). Kolmogorov forward equation and explosiveness in countable state Markov processes. *Ann. Oper. Res.*, to appear. DOI: 10.1007/s10479-012-1262-7.
- [31] Spieksma, F. (2015). Countable state Markov processes: non-explosiveness and moment function. *Probab. Eng. Inform. Sc.*, **29**, 623-637.
- [32] Wagner, W. (2005). Explosion phenomena in stochastic coagulation-gragmentation models. *Ann. Appl. Probab.*, **15**, 2081-2112.
- [33] Ye, L. and Guo, X. (2013). Construction and regularity of transition functions on Polish spaces under measurability conditions. *Acta Math. Appl. Sin.* (English Series), **29**, 1-14.
- [34] Yushkevich, A. (1980). On reducing a jump controllable Markov model to a model with discrete time. *Theory Probab. Appl.*, **25**, 58-69.
- [35] Zheng, J. and Zheng, X. (1987). A martingale approach to  $q$ -processes. *Kexue Tongbao (Chinese Science Bulletin)*, **32**, 1457-1459.
- [36] Zheng, J. 1993. *Phase Transitions of Ising Model on Lattice Fractals, Martingale Approach for  $q$ -processes* (in Chinese). Ph.D. Thesis. Beijing Normal University.